



Solution of Triangle

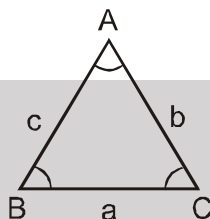
According to most accounts, geometry was first discovered among the Egyptians, taking its origin from the measurement of areas. For they found it necessary by reason of the flooding of the Nile, which wiped out everybody's proper boundaries. Nor is there anything surprising in that the discovery both of this and of the other sciences should have had its origin in a practical need, since everything which is in process of becoming progresses from the imperfect to the perfect.

.....Proclus

Sine Rule :

In any triangle ABC, the sines of the angles are proportional to the opposite sides

$$\text{i.e. } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



Example # 1 : How many triangles can be constructed with the data : $a = 5$, $b = 7$, $\sin A = 3/4$

Solution : Since $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{5}{3/4} = \frac{7}{\sin B}$
 $\Rightarrow \sin B = \frac{21}{20} > 1$ not possible
 $\therefore$ no triangle can be constructed.

Example # 2 : If in a triangle ABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, then show that a^2, b^2, c^2 are in A.P.

Solution : We have $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$
 $\Rightarrow \sin(B+C) \sin(B-C) = \sin(A+B) \sin(A-B) \Rightarrow \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$
 $\Rightarrow b^2 - c^2 = a^2 - b^2 \Rightarrow a^2, b^2, c^2$ are in A.P.

Self Practice Problems :

- (1) In a $\triangle ABC$, the sides a, b and c are in A.P., then prove that $\left(\tan \frac{A}{2} + \tan \frac{C}{2}\right) : \cot \frac{B}{2} = 2 : 3$
- (2) If the angles of $\triangle ABC$ are in the ratio $1 : 2 : 3$, then find the ratio of their corresponding sides
- (3) In a $\triangle ABC$ prove that $\frac{c}{a-b} = \frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{\tan \frac{A}{2} - \tan \frac{B}{2}}$.

Ans. (2) $1 : \sqrt{3} : 2$

Cosine Formula :

In any $\triangle ABC$

$$(i) \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{or} \quad a^2 = b^2 + c^2 - 2bc \cos A = b^2 + c^2 + 2bc \cos(B+C)$$

$$(ii) \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$(iii) \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$





Example # 3 : In a triangle ABC, A, B, C are in A.P. Show that $2\cos\left(\frac{A-C}{2}\right) = \frac{a+c}{\sqrt{a^2-ac+c^2}}$.

Solution : $A + C = 2B \Rightarrow A + B + C = 3B \Rightarrow B = 60^\circ$

$$\therefore \cos 60^\circ = \frac{a^2 + c^2 - b^2}{2ac} \Rightarrow a^2 - ac + c^2 = b^2$$

$$\begin{aligned} \Rightarrow \frac{a+c}{\sqrt{a^2-ac+c^2}} &= \frac{a+c}{b} = \left[\frac{\sin A + \sin C}{\sin B} \right] = \frac{2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)}{\sin B} \\ &= 2\cos\frac{A-C}{2} \quad (\because A+C=2B) \end{aligned}$$

Example # 4 : In a $\triangle ABC$, prove that $a(b\cos C - c\cos B) = b^2 - c^2$

Solution : Since $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ & $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

$$\begin{aligned} \therefore \text{L.H.S.} &= a \left\{ b \left(\frac{a^2 + b^2 - c^2}{2ab} \right) - c \left(\frac{a^2 + c^2 - b^2}{2ac} \right) \right\} \\ &= \frac{a^2 + b^2 - c^2}{2} - \frac{(a^2 + c^2 - b^2)}{2} = (b^2 - c^2) = \text{R.H.S.} \end{aligned}$$

Hence L.H.S. = R.H.S.

Proved

Example # 5 : The sides of $\triangle ABC$ are $AB = \sqrt{13}$ cm, $BC = 4\sqrt{3}$ cm and $CA = 7$ cm. Then find the value of $\sin\theta$ where θ is the smallest angle of the triangle.

Solution : Angle opposite to AB is smallest. Therefore,

$$\cos\theta = \frac{49 + 48 - 13}{2 \cdot 7 \cdot 4\sqrt{3}} = \frac{\sqrt{3}}{2} \Rightarrow \sin\theta = \frac{1}{2}$$

Self Practice Problems :

(4) If in a triangle ABC, $3\sin A = 6\sin B = 2\sqrt{3}\sin C$, Then find the angle A.

(5) If two sides a, b and angle A be such that two triangles are formed, then find the sum of two values of the third side.

Ans. (4) 90° (5) $2b\cos A$

Projection Formula :

In any $\triangle ABC$

(i) $a = b\cos C + c\cos B$

(ii) $b = c\cos A + a\cos C$

(iii) $c = a\cos B + b\cos A$

Example # 6 : If in a $\triangle ABC$, $c\cos^2\frac{A}{2} + a\cos^2\frac{C}{2} = \frac{3b}{2}$, then show that a, b, c are in A.P.

Solution : $c(1 + \cos A) + a(1 + \cos C) = 3b$
 $\Rightarrow a + c + (c\cos A + a\cos C) = 3b$
 $\Rightarrow a + c + b = 3b$
 $\Rightarrow a + c = 2b$

Example # 7 : In a $\triangle ABC$, prove that $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c$.

Solution : $\therefore \text{L.H.S.} = (b+c)\cos A + (c+a)\cos B + (a+b)\cos C$
 $= b\cos A + c\cos A + c\cos B + a\cos B + a\cos C + b\cos C$
 $= (b\cos A + a\cos B) + (c\cos A + a\cos C) + (c\cos B + b\cos C)$
 $= a + b + c$
 $= \text{R.H.S.}$

Hence L.H.S. = R.H.S.

Proved





Self Practice Problems :

- (6) The roots of $x^2 - 2\sqrt{3}x + 2 = 0$ represent two sides of a triangle. If the angle between them is $\frac{\pi}{3}$, then find the perimeter of triangle.
- (7) In a triangle ABC, if $\cos A + \cos B + \cos C = \frac{3}{2}$, then show that the triangle is an equilateral triangle.
- (8) In a $\triangle ABC$, prove that $\frac{\cos A}{c \cos B + b \cos C} + \frac{\cos B}{a \cos C + c \cos A} + \frac{\cos C}{a \cos B + b \cos A} = \frac{a^2 + b^2 + c^2}{2abc}$.

Ans. (6) $2\sqrt{3} + \sqrt{6}$

Napier's Analogy - tangent rule :

In any $\triangle ABC$

$$(i) \quad \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} \quad (ii) \quad \tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2}$$

$$(iii) \quad \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$$

Example # 8 : Find the unknown elements of the $\triangle ABC$ in which $a = \sqrt{3} + 1$, $b = \sqrt{3} - 1$, $C = 90^\circ$.

Solution :

$$\begin{aligned} \therefore a &= \sqrt{3} + 1, b = \sqrt{3} - 1, C = 90^\circ \\ \therefore A + B + C &= 180^\circ \\ \therefore A + B &= 90^\circ \quad \dots\dots(i) \\ \therefore \text{From law of tangent, we know that } \tan \left(\frac{A-B}{2} \right) &= \frac{a-b}{a+b} \cot \frac{C}{2} \\ &= \frac{(\sqrt{3}+1) - (\sqrt{3}-1)}{(\sqrt{3}+1) + (\sqrt{3}-1)} \cot 45^\circ = \frac{2}{2\sqrt{3}} \cot 45^\circ \Rightarrow \tan \left(\frac{A-B}{2} \right) = \frac{1}{\sqrt{3}} \\ \therefore \frac{A-B}{2} &= \frac{\pi}{6} \\ \Rightarrow A - B &= \frac{\pi}{3} \quad \dots\dots(ii) \end{aligned}$$

From equation (i) and (ii), we get $A = \frac{5\pi}{12}$ and $B = \frac{\pi}{12}$

Now, $c = \sqrt{a^2 + b^2} = 2\sqrt{2}$

$\therefore c = 2\sqrt{2}, A = \frac{5\pi}{12}, B = \frac{\pi}{12}$ **Ans.**

Self Practice Problems :

- (9) In a $\triangle ABC$ if $b = 3$, $c = 5$ and $\cos(B - C) = \frac{7}{25}$, then find the value of $\sin \frac{A}{2}$.
- (10) If in a $\triangle ABC$, we define $x = \tan \left(\frac{B-C}{2} \right) \tan \frac{A}{2}$, $y = \tan \left(\frac{C-A}{2} \right) \tan \frac{B}{2}$ and $z = \tan \left(\frac{A-B}{2} \right) \tan \frac{C}{2}$, then show that $x + y + z = -xyz$.

Ans. (9) $\frac{1}{\sqrt{10}}$



Trigonometric Functions of Half Angles :

- (i) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$, $\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$
- (ii) $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$, $\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$
- (iii) $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)} = \frac{(s-b)(s-c)}{\Delta}$, where $s = \frac{a+b+c}{2}$ is semi perimeter and Δ is the area of triangle.
- (iv) $\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$

Area of Triangle (Δ)

$$\Delta = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \sqrt{s(s-a)(s-b)(s-c)}$$

Example # 9 : If p_1, p_2, p_3 are the altitudes of a triangle ABC from the vertices A, B, C and Δ is the area of the triangle, then show that $p_1^{-1} + p_2^{-1} + p_3^{-1} = \frac{s-c}{\Delta}$

Solution : We have

$$\begin{aligned} \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} &= \frac{a}{2\Delta} + \frac{b}{2\Delta} + \frac{c}{2\Delta} \\ &= \frac{a+b+c}{2\Delta} = \frac{2(s-c)}{2\Delta} = \frac{s-c}{\Delta} \end{aligned}$$

Example # 10 : In a ΔABC if $b \sin C (b \cos C + c \cos B) = 64$, then find the area of the ΔABC .

Solution : $\therefore b \sin C (b \cos C + c \cos B) = 64$ (i) given

$\therefore$ From **projection rule**, we know that
 $a = b \cos C + c \cos B$ put in (i), we get
 $ab \sin C = 64$ (ii)

$\therefore \Delta = \frac{1}{2} ab \sin C$ $\therefore$ from equation (ii), we get

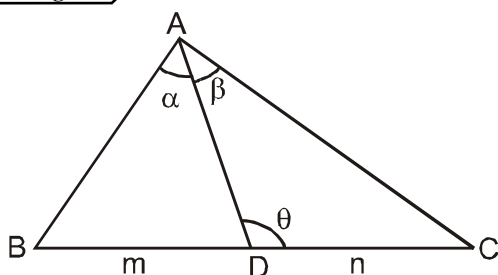
$\therefore \Delta = 32$ sq. unit

Example # 11 : If A,B,C are the angle of a triangle, then prove that $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \frac{s^2}{\Delta}$

Solution :

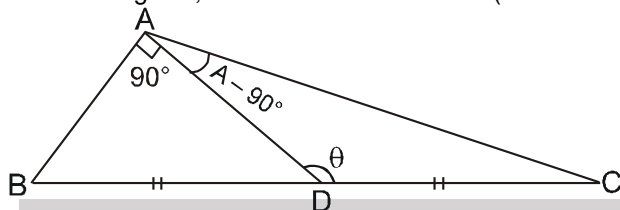
$$\begin{aligned} &\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} \\ &= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-c)(s-a)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} \\ &= \frac{\sqrt{s(s-a+s-b+s-c)}}{\sqrt{(s-a)(s-b)(s-c)}} = \frac{s}{\Delta} (3s-2s) = \frac{s^2}{\Delta} \end{aligned}$$

m - n Rule : In any triangle ABC if D be any point on the base BC, such that $BD : DC :: m : n$ and if $\angle BAD = \alpha$, $\angle DAC = \beta$, $\angle CDA = \theta$, then
 $(m+n) \cot \theta = m \cot \alpha + n \cot \beta$
 $n \cot B + m \cot C$



Example # 12 : In a $\triangle ABC$. AD divides BC in the ratio 2 : 1 such that at $\angle BAD = 90^\circ$ then prove that $\tan A + 3 \tan B = 0$

Solution : From the figure, we see that $\theta = 90^\circ + B$ (as θ is external angle of $\triangle ABD$)

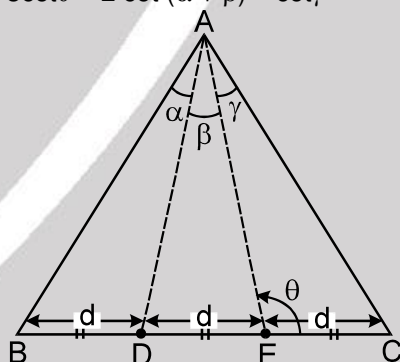


Now if we apply **m-n rule** in $\triangle ABC$, we get
 $(2 + 1) \cot (90^\circ + B) = 2 \cdot \cot 90^\circ - 1 \cdot \cot (A - 90^\circ)$
 $\Rightarrow -3 \tan B = \cot (90^\circ - A)$
 $\Rightarrow -3 \tan B = \tan A$
 $\Rightarrow \tan A + 3 \tan B = 0$ **Hence proved.**

Example # 13 : The base of a $\triangle$ is divided into three equal parts. If α, β, γ be the angles subtended by these parts at the vertex, prove that :

Solution : Let point D and E divides the base BC into three equal parts i.e. $BD = DE = EC = d$ (Let) and let α, β and γ be the angles subtended by BD, DE and EC respectively at their opposite vertex. Now in $\triangle ABC$

$\therefore BE : EC = 2d : d = 2 : 1$
 $\therefore$ from **m-n rule**, we get
 $(2 + 1) \cot \theta = 2 \cot (\alpha + \beta) - \cot \gamma$
 $\Rightarrow 3 \cot \theta = 2 \cot (\alpha + \beta) - \cot \gamma$ (i)



again

$\therefore$ in $\triangle ADC$
 $\therefore DE : EC = d : d = 1 : 1$
 $\therefore$ if we apply **m-n rule** in $\triangle ADC$, we get
 $(1 + 1) \cot \theta = 1 \cdot \cot \beta - 1 \cdot \cot \gamma$
 $2 \cot \theta = \cot \beta - \cot \gamma$ (ii)

from (i) and (ii), we get $\frac{3 \cot \theta}{2 \cot \theta} = \frac{2 \cot (\alpha + \beta) - \cot \gamma}{\cot \beta - \cot \gamma}$

$\Rightarrow 3 \cot \beta - 3 \cot \gamma = 4 \cot (\alpha + \beta) - 2 \cot \gamma$





$$\begin{aligned}
 &\Rightarrow 3\cot\beta - \cot\gamma = 4 \cot(\alpha + \beta) \\
 &\Rightarrow 3\cot\beta - \cot\gamma = 4 \left\{ \frac{\cot\alpha \cot\beta - 1}{\cot\beta + \cot\alpha} \right\} \\
 &\Rightarrow 3\cot^2\beta + 3\cot\alpha \cot\beta - \cot\beta \cot\gamma - \cot\alpha \cot\gamma = 4 \cot\alpha \cot\beta - 4 \\
 &\Rightarrow 4 + 3\cot^2\beta = \cot\alpha \cot\beta + \cot\beta \cot\gamma + \cot\alpha \cot\gamma \\
 &\Rightarrow 4 + 4\cot^2\beta = \cot\alpha \cot\beta + \cot\alpha \cot\gamma + \cot\beta \cot\gamma + \cot^2\beta \\
 &\Rightarrow 4(1 + \cot^2\beta) = (\cot\alpha + \cot\beta)(\cot\beta + \cot\gamma) \\
 &\Rightarrow (\cot\alpha + \cot\beta)(\cot\beta + \cot\gamma) = 4\operatorname{cosec}^2\beta
 \end{aligned}$$

Self Practice Problems :

- (11) In a $\triangle ABC$, the median to the side BC is of length $\frac{1}{\sqrt{11-6\sqrt{3}}}$ unit and it divides angle A into the angles of 30° and 45° . Prove that the side BC is of length 2 unit.

Radius of Circumcircle :

If R be the circumradius of $\triangle ABC$, then $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$

Example # 14 : In a $\triangle ABC$, prove that $\sin 2A + \sin 2B + \sin 2C = 2\Delta/R^2$

Solution : In a $\triangle ABC$, we know that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$
 and $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$
 $= \frac{4abc}{8R^3} = \frac{16\Delta R}{8R^3} = \frac{2\Delta}{R^2}$

Example # 15 : In a $\triangle ABC$ if $a = 22$ cm, $b = 28$ cm and $c = 36$ cm, then find its circumradius.

Solution : $\therefore R = \frac{abc}{4\Delta} \dots\dots(i)$
 $\therefore \Delta = \sqrt{s(s-a)(s-b)(s-c)}$
 $\therefore s = \frac{a+b+c}{2} = 43$ cm
 $\therefore \Delta = \sqrt{43 \times 21 \times 15 \times 7} = 21\sqrt{215}$
 $\therefore R = \frac{22 \times 28 \times 36}{4 \times 21\sqrt{215}} = \frac{264}{\sqrt{215}}$ cm

Example # 16 : In a $\triangle ABC$, if $8R^2 = a^2 + b^2 + c^2$, show that the triangle is right angled.

Solution : We have : $8R^2 = a^2 + b^2 + c^2$
 $\Rightarrow 8R^2 = [4R^2 \sin^2 A + 4R^2 \sin^2 B + 4R^2 \sin^2 C] \quad [\because a = 2R \sin A \text{ etc.}]$
 $\Rightarrow 2 = \sin^2 A + \sin^2 B + \sin^2 C \Rightarrow (1 - \sin^2 A) - \sin^2 B + (1 - \sin^2 C) = 0$
 $\Rightarrow (\cos^2 A - \sin^2 B) + \cos^2 C = 0 \Rightarrow \cos(A+B) \cos(A-B) + \cos^2 C = 0$
 $\Rightarrow -\cos C \cos(A-B) + \cos^2 C = 0 \Rightarrow -\cos C \{\cos(A-B) - \cos C\} = 0$
 $\Rightarrow -\cos C [\cos(A-B) + \cos(A+B)] = 0 \Rightarrow -2\cos A \cos B \cos C = 0$
 $\Rightarrow \cos A = 0 \text{ or } \cos B = 0 \text{ or } \cos C = 0$
 $\Rightarrow A = \frac{\pi}{2} \text{ or } B = \frac{\pi}{2} \text{ or } C = \frac{\pi}{2}$
 $\Rightarrow \triangle ABC$ is a right angled triangle.

Example # 17 : $\frac{b^2 - c^2}{2a} = R \sin(B - C)$

Solution : $\frac{b^2 - c^2}{2a} = \frac{4R^2(\sin^2 B - \sin^2 C)}{4R \sin A} = \frac{R \sin(B+C) \sin(B-C)}{\sin A} = R \sin(B - C)$



Self Practice Problems :

(12) In a $\triangle ABC$, prove that $(a + b) = 4R \cos\left(\frac{A-B}{2}\right) \cos \frac{C}{2}$

(13) In a $\triangle ABC$, if $b = 15$ cm and $\cos B = \frac{4}{5}$, find R .

(14) In a triangle ABC if α, β, γ are the distances of the vertices of triangle from the corresponding points of contact with the incircle, then prove that $\frac{\alpha\beta\gamma}{\alpha + \beta + \gamma} = r^2$

Ans. (13) 12.5

Radius of The Incircle :

If 'r' be the inradius of $\triangle ABC$, then

(i) $r = \frac{\Delta}{s}$

(ii) $r = (s-a) \tan \frac{A}{2} = (s-b) \tan \frac{B}{2} = (s-c) \tan \frac{C}{2}$

(iii) $r = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}}$ and so on

(iv) $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

Radius of The Ex-Circles :

If r_1, r_2, r_3 are the radii of the ex-circles of $\triangle ABC$ opposite to the vertex A, B, C respectively, then

(i) $r_1 = \frac{\Delta}{s-a}; r_2 = \frac{\Delta}{s-b}; r_3 = \frac{\Delta}{s-c};$

(ii) $r_1 = s \tan \frac{A}{2}; r_2 = s \tan \frac{B}{2}; r_3 = s \tan \frac{C}{2}$

(iii) $r_1 = \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}$ and so on

(iv) $r_1 = 4R \sin \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}$

Example # 18 : $\cos A + \cos B + \cos C = \left(1 + \frac{r}{R}\right)$

Solution :

$$\begin{aligned} \text{LHS} &= \cos A + \cos B + \cos C \\ &= 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) + 1 - 2 \sin^2 \frac{C}{2} \\ &= 2 \sin \frac{C}{2} \left\{ \cos\left(\frac{A-B}{2}\right) - \sin \frac{C}{2} \right\} + 1 = 2 \sin \frac{C}{2} \left\{ \cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right) \right\} + 1 \\ &= 2 \sin \frac{C}{2} \left\{ 2 \sin \frac{A}{2} \sin \frac{B}{2} \right\} + 1 = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\ &= 1 + \frac{1}{R} \left(4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) = 1 + \frac{r}{R} = \text{RHS} \end{aligned}$$

Example # 19 : In a triangle ABC , find the value of $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3}$.

Solution :

$$\begin{aligned} &\frac{b-c}{\left(\frac{\Delta}{s-a}\right)} + \frac{c-a}{\left(\frac{\Delta}{s-b}\right)} + \frac{a-b}{\left(\frac{\Delta}{s-c}\right)} \\ &= \frac{1}{\Delta} [(b-c)(s-a) + (c-a)(s-b) + (a-b)(s-c)] \end{aligned}$$



$$= \frac{1}{\Delta} [s(b-c+c-a+a-b) - a(b-c) - b(c-a) - c(a-b)] = 0$$

Self Practice Problems :

- (15) In a triangle ABC, r_1, r_2, r_3 are in HP. If its area is 24 cm^2 and its perimeter is 24 cm. then find lengths of its sides.
- (16) In a triangle ABC, $a : b : c = 4 : 5 : 6$. Find the ratio of the radius of the circumcircle to that of the incircle.
- (17) In a ΔABC , prove that $\frac{r_1 - r}{a} + \frac{r_2 - r}{b} = \frac{c}{r_3}$.
- (18) If A, A_1, A_2 and A_3 are the areas of the inscribed and escribed circles respectively of a ΔABC , then prove that $\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$.

Ans. (15) 6, 8, 10 (16) 16 : 7

Length of Angle Bisectors, Medians & Altitudes :

- (i) Length of an angle bisector from the angle $A = \beta_a = \frac{2bc \cos \frac{A}{2}}{b+c}$;
- (ii) Length of median from the angle $A = m_a = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}$
- & (iii) Length of altitude from the angle $A = A_a = \frac{2\Delta}{a}$

NOTE : $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4} (a^2 + b^2 + c^2)$

Example #20 : In ΔABC , AD & BE are its two median. If $AD = 4$, $\angle DAB = \frac{\pi}{6}$ and $\angle ABE = \frac{\pi}{3}$ then find the length of BE and area of ΔABC .

Solution : $AP = \frac{2}{3}$; $AD = \frac{8}{3}$; $PD = \frac{4}{3}$; Let $PB = x$

$$\tan 60^\circ = \frac{8/3}{x} \quad \text{or} \quad x = \frac{8}{3\sqrt{3}}$$

$$\text{Area of } \Delta ABP = \frac{1}{2} \times \frac{8}{3} \times \frac{8}{3\sqrt{3}} = \frac{32}{9\sqrt{3}}$$

$$\therefore \text{Area of } \Delta ABC = 3 \times \frac{32}{9\sqrt{3}} = \frac{32}{3\sqrt{3}}$$

$$\text{Also, } BE = \frac{3}{2}x = \frac{4}{\sqrt{3}}$$

Self Practice Problem :

- (19) In a ΔABC if $\angle A = 90^\circ$, $b = 5 \text{ cm}$, $c = 12 \text{ cm}$. If 'G' is the centroid of triangle, then find circumradius of ΔGAB .



Ans. (19) $\frac{13\sqrt{601}}{30}$ cm

The Distances of The Special Points from Vertices and Sides of Triangle :

- (i) Circumcentre (O) : $OA = R$ and $O_a = R \cos A$
- (ii) Incentre (I) : $IA = r \operatorname{cosec} \frac{A}{2}$ and $I_a = r$
- (iii) Excentre (I_1) : $I_1 A = r_1 \operatorname{cosec} \frac{A}{2}$ and $I_{1a} = r_1$
- (iv) Orthocentre (H) : $HA = 2R \cos A$ and $H_a = 2R \cos B \cos C$
- (v) Centroid (G) : $GA = \frac{1}{3} \sqrt{2b^2 + 2c^2 - a^2}$ and $G_a = \frac{2\Delta}{3a}$

Example # 21 : If p_1, p_2, p_3 are respectively the lengths of perpendiculars from the vertices of a triangle ABC to the opposite sides, prove that :

(i) $\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3} = \frac{1}{R}$ (ii) $\frac{bp_1}{c} + \frac{cp_2}{a} + \frac{ap_3}{b} = \frac{a^2 + b^2 + c^2}{2R}$

Solution :

(i) use $\frac{1}{p_1} = \frac{a}{2\Delta}, \frac{1}{p_2} = \frac{b}{2\Delta}, \frac{1}{p_3} = \frac{c}{2\Delta}$

$$\begin{aligned} \therefore \text{LHS} &= \frac{1}{2\Delta} (a \cos A + b \cos B + c \cos C) \\ &= \frac{R}{2\Delta} (\sin 2A + \sin 2B + \sin 2C) = \frac{4R \sin A \sin B \sin C}{2\Delta} \\ &= \frac{4R}{2\Delta} \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{1}{4\Delta R^2} abc = \frac{1}{4\Delta R^2} \cdot (4R\Delta) = \frac{1}{R} = \text{RHS} \end{aligned}$$

(ii) $\text{LHS} = \frac{bp_1}{c} + \frac{cp_2}{a} + \frac{ap_3}{b} = \frac{a^2 + b^2 + c^2}{2R} = \frac{2b\Delta}{ac} + \frac{2c\Delta}{ab} + \frac{2a\Delta}{bc} = \frac{2\Delta(a^2 + b^2 + c^2)}{abc}$

$$= \frac{2\Delta(a^2 + b^2 + c^2)}{4\Delta R} = \frac{a^2 + b^2 + c^2}{2R}$$

Self Practice Problems :

(20) If I be the incentre of $\triangle ABC$, then prove that $IA \cdot IB \cdot IC = abc \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}$.

(21) If x, y, z are respectively be the perpendiculars from the circumcentre to the sides of $\triangle ABC$, then prove that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$.



Exercise-1

✎ Marked questions are recommended for Revision.

SUBJECTIVE QUESTIONS

Section (A) : Sine rule, Cosine rule, Napier's Analogy, Projection rule

A-1. In a $\triangle ABC$, prove that :

- (i) $a \sin(B - C) + b \sin(C - A) + c \sin(A - B) = 0$
- (ii) $\frac{a^2 \sin(B - C)}{\sin A} + \frac{b^2 \sin(C - A)}{\sin B} + \frac{c^2 \sin(A - B)}{\sin C} = 0$
- (iii) $2(bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$
- (iv) ✎ $(a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2} = c^2$
- (v) $b^2 \sin 2C + c^2 \sin 2B = 2bc \sin A$
- (vi) $\frac{\sin B}{\sin C} = \frac{c - a \cos B}{b - a \cos C}$

A-2. Find the real value of x such that $x^2 + 2x$, $2x + 3$ and $x^2 + 3x + 8$ are lengths of the sides of a triangle.

A-3. The angles of a $\triangle ABC$ are in A.P. (order being A, B, C) and it is being given that $b : c = \sqrt{3} : \sqrt{2}$, then find $\angle A$.

A-4. ✎ If $\cos A + \cos B = 4 \sin^2 \left(\frac{C}{2} \right)$, prove that sides a, c, b of the triangle ABC are in A.P.

A-5. If in a $\triangle ABC$, $\frac{\sin A}{\sin C} = \frac{\sin(A - B)}{\sin(B - C)}$, then prove that a^2, b^2, c^2 are in A.P.

A-6. In a triangle ABC , prove that for any angle θ , $b \cos(A - \theta) + a \cos(B + \theta) = c \cos \theta$.

A-7. ✎ With usual notations, if in a $\triangle ABC$, $\frac{b + c}{11} = \frac{c + a}{12} = \frac{a + b}{13}$, then prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$.

A-8. Let a, b and c be the sides of a $\triangle ABC$. If a^2, b^2 and c^2 are the roots of the equation $x^3 - Px^2 + Qx - R = 0$, where P, Q & R are constants, then find the value of $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$ in terms of P, Q and R .

A-9. ✎ If in a triangle ABC , the altitude AM be the bisector of $\angle BAD$, where D is the mid point of side BC , then prove that $(b^2 - c^2) = a^2/2$.

A-10. If in a triangle ABC , $\angle C = 60^\circ$, then prove that $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

A-11. ✎ In a triangle ABC , $\angle C = 60^\circ$ and $\angle A = 75^\circ$. If D is a point on AC such that the area of the $\triangle ABD$ is $\sqrt{3}$ times the area of the $\triangle BCD$, find the $\angle ABD$.

A-12. ✎ In a scalene triangle ABC , D is a point on the side AB such that $CD^2 = AD \cdot DB$, if $\sin A \cdot \sin B = \sin^2 \frac{C}{2}$ then prove that CD is internal bisector of $\angle C$.

A-13. ✎ In triangle ABC , D is on AC such that $AD = BC$, $BD = DC$, $\angle DBC = 2x$, and $\angle BAD = 3x$, all angles are in degrees, then find the value of x .





Section (B) Trigonometric ratios of Half Angles, Area of triangle and circumradius

B-1. In a $\triangle ABC$, prove that

$$(i) \quad 2 \left[a \sin^2 \frac{C}{2} + c \sin^2 \frac{A}{2} \right] = c + a - b.$$

$$(ii) \quad \frac{\cos^2 \frac{A}{2}}{a} + \frac{\cos^2 \frac{B}{2}}{b} + \frac{\cos^2 \frac{C}{2}}{c} = \frac{s^2}{abc}$$

$$(iii) \quad 4 \left(bc \cos^2 \frac{A}{2} + ca \cos^2 \frac{B}{2} + ab \cos^2 \frac{C}{2} \right) = (a + b + c)^2$$

$$(iv) \quad (b - c) \cot \frac{A}{2} + (c - a) \cot \frac{B}{2} + (a - b) \cot \frac{C}{2} = 0$$

$$(v) \quad 4\Delta (\cot A + \cot B + \cot C) = a^2 + b^2 + c^2$$

$$(vi) \quad \left(\frac{2abc}{a + b + c} \right) \cdot \cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2} = \Delta$$

B-2. If the sides a, b, c of a triangle are in A.P., then find the value of $\tan \frac{A}{2} + \tan \frac{C}{2}$ in terms of $\cot(B/2)$.

B-3. If in a $\triangle ABC$, $a = 6, b = 3$ and $\cos(A - B) = 4/5$, then find its area.

B-4. If in a triangle ABC , $\angle A = 30^\circ$ and the area of triangle is $\frac{\sqrt{3} a^2}{4}$, then prove that either $B = 4C$ or $C = 4B$.

Section (C) Inradius and Exradius

C-1. In any $\triangle ABC$, prove that

$$(i) \quad Rr (\sin A + \sin B + \sin C) = \Delta$$

$$(ii) \quad a \cos B \cos C + b \cos C \cos A + c \cos A \cos B = \frac{\Delta}{R}$$

$$(iii) \quad \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}.$$

$$(iv) \quad \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = 2 + \frac{r}{2R}$$

$$(v) \quad a \cot A + b \cot B + c \cot C = 2(R + r)$$

C-2. In any $\triangle ABC$, prove that

$$(i) \quad r_1 \cdot r_2 \cdot r_3 = \Delta^2$$

$$(ii) \quad r_1 + r_2 - r_3 + r = 4R \cos C.$$

$$(iii) \quad \frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2 + b^2 + c^2}{\Delta^2}$$

$$(iv) \quad \left(\frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right)^2 = \frac{4}{r} \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right)$$

$$(v) \quad \frac{bc - r_2 r_3}{r_1} = \frac{ca - r_3 r_1}{r_2} = \frac{ab - r_1 r_2}{r_3} = r$$



- C-3.** Show that the radii of the three escribed circles of a triangle are roots of the equation $x^3 - x^2(4R + r) + x s^2 - r s^2 = 0$.
- C-4.** The radii r_1, r_2, r_3 of escribed circles of a triangle ABC are in harmonic progression. If its area is 24 sq. cm and its perimeter is 24 cm, find the lengths of its sides.
- C-5.** If the area of a triangle is 100 sq.cm, $r_1 = 10$ cm and $r_2 = 50$ cm, then find the value of $(b - a)$.

Section (D) Miscellaneous

- D-1.** If α, β, γ are the respective altitudes of a triangle ABC, prove that
- (i)
$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\cot A + \cot B + \cot C}{\Delta}$$
- (ii)
$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{2ab}{(a + b + c) \Delta} \cos^2 \frac{C}{2}$$
- D-2.** If in an acute angled $\triangle ABC$, line joining the circumcentre and orthocentre is parallel to side AC, then find the value of $\tan A \cdot \tan C$.
- D-3.** A regular hexagon & a regular dodecagon are inscribed in the same circle. If the side of the dodecagon is $(\sqrt{3} - 1)$, if the side of the hexagon is $\sqrt[4]{k}$, then find value of k.
- D-4.** If D is the mid point of CA in triangle ABC and Δ is the area of triangle, then show that
$$\tan(\angle ADB) = \frac{4\Delta}{a^2 - c^2}.$$

Exercise-2

Marked questions are recommended for Revision.

PART-I (OBJECTIVE QUESTIONS)

Section (A) : Sine rule, Cosine rule, Napier's Analogy, Projection rule

- A-1.** In a $\triangle ABC$, $A : B : C = 3 : 5 : 4$. Then $a + b + c \sqrt{2}$ is equal to
(A) $2b$ (B) $2c$ (C) $3b$ (D) $3a$
- A-2*.** In a triangle ABC, the altitude from A is not less than BC and the altitude from B is not less than AC. The triangle is
(A) right angled (B) isosceles (C) obtuse angled (D) equilateral
- A-3.** If in a $\triangle ABC$, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, then the triangle is :
(A) right angled (B) isosceles (C) equilateral (D) obtuse angled
- A-4.** In a $\triangle ABC$ $\frac{bc \sin^2 A}{\cos A + \cos B \cos C}$ is equal to
(A) $b^2 + c^2$ (B) bc (C) a^2 (D) $a^2 + bc$
- A-5.** Given a triangle $\triangle ABC$ such that $\sin^2 A + \sin^2 C = 1001 \sin^2 B$. Then the value of $\frac{2(\tan A + \tan C) \cdot \tan^2 B}{\tan A + \tan B + \tan C}$ is
(A) $\frac{1}{2000}$ (B) $\frac{1}{1000}$ (C) $\frac{1}{500}$ (D) $\frac{1}{250}$





- A-6.** If in a triangle ABC, $(a + b + c)(b + c - a) = k \cdot bc$, then :
 (A) $k < 0$ (B) $k > 6$ (C) $0 < k < 4$ (D) $k > 4$
- A-7.** In a triangle ABC, $a : b : c = 4 : 5 : 6$. Then $3A + B$ equals to :
 (A) $4C$ (B) 2π (C) $\pi - C$ (D) π
- A-8.** The distance between the middle point of BC and the foot of the perpendicular from A is ($b > c$) :
 (A) $\frac{-a^2 + b^2 + c^2}{2a}$ (B) $\frac{b^2 - c^2}{2a}$ (C) $\frac{b^2 + c^2}{\sqrt{bc}}$ (D) $\frac{b^2 + c^2}{2a}$
- A-9*.** If in a triangle ABC, $\cos A \cos B + \sin A \sin B \sin C = 1$, then the triangle is
 (A) isosceles (B) right angled (C) equilateral (D) None of these
- A-10.** Triangle ABC is right angle at A. The points P and Q are on hypotenuse BC such that $BP = PQ = QC$. If $AP = 3$ and $AQ = 4$, then length BC is equal to
 (A) $3\sqrt{5}$ (B) $5\sqrt{3}$ (C) $4\sqrt{5}$ (D) 7
- A-11.** In $\triangle ABC$, $bc = 2b^2 \cos A + 2c^2 \cos A - 4bc \cos^2 A$, then $\triangle ABC$ is
 (A) isosceles but not necessarily equilateral
 (B) equilateral
 (C) right angled but not necessarily isosceles
 (D) right angled isosceles

Section (B) Trigonometric ratios of Half Angles, Area of triangle and circumradius

- B-1.** If in a triangle ABC, right angle at B, $s - a = 3$ and $s - c = 2$, then
 (A) $a = 2, c = 3$ (B) $a = 3, c = 4$ (C) $a = 4, c = 3$ (D) $a = 6, c = 8$
- B-2.** If in a triangle ABC, $b \cos^2 \frac{A}{2} + a \cos^2 \frac{B}{2} = \frac{3}{2}c$, then a, c, b are :
 (A) in A.P. (B) in G.P. (C) in H.P. (D) None
- B-3.** If H is the orthocentre of a triangle ABC, then the radii of the circle circumscribing the triangles BHC, CHA and AHB are respectively equal to :
 (A) R, R, R (B) $\sqrt{2}R, \sqrt{2}R, \sqrt{2}R$ (C) $2R, 2R, 2R$ (D) $\frac{R}{2}, \frac{R}{2}, \frac{R}{2}$
- B-4.** In a $\triangle ABC$ if $b + c = 3a$, then $\cot \frac{B}{2} \cdot \cot \frac{C}{2}$ has the value equal to:
 (A) 4 (B) 3 (C) 2 (D) 1
- B-5.** In a $\triangle ABC$, $A = \frac{2\pi}{3}$, $b - c = 3\sqrt{3}$ cm and area ($\triangle ABC$) = $\frac{9\sqrt{3}}{2}$ cm². Then 'a' is
 (A) $6\sqrt{3}$ cm (B) 9 cm (C) 18 cm (D) 7 cm
- B-6.*** The diagonals of a parallelogram are inclined to each other at an angle of 45° , while its sides a and b ($a > b$) are inclined to each other at an angle of 30° , then the value of $\frac{a}{b}$ is
 (A) $2\cos 36^\circ$ (B) $\sqrt{\frac{3+\sqrt{5}}{4}}$ (C) $\frac{3+\sqrt{5}}{4}$ (D) $\frac{\sqrt{5}+1}{2}$





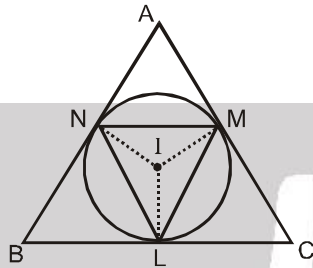
- B-7.** If in a $\triangle ABC$, $\Delta = a^2 - (b - c)^2$, then $\tan A$ is equal to
 (A) 15/16 (B) 8/15 (C) 8/17 (D) 1/2
- B-8*.** If in a $\triangle ABC$, $a = 5$, $b = 4$ and $\cos(A - B) = \frac{31}{32}$, then
 (A) $c = 6$ (B) $\sin A = \left(\frac{5\sqrt{7}}{16}\right)$
 (C) area of $\triangle ABC = \frac{15\sqrt{7}}{4}$ (D) $c = 8$
- B-9.** If R denotes circumradius, then in $\triangle ABC$, $\frac{b^2 - c^2}{2aR}$ is equal to
 (A) $\cos(B - C)$ (B) $\sin(B - C)$ (C) $\cos B - \cos C$ (D) $\sin(B + C)$
- B-10*.** Which of the following holds good for any triangle ABC ?
 (A) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$ (B) $\frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c} = \frac{3}{2R}$
 (C) $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ (D) $\frac{\sin 2A}{a^2} = \frac{\sin 2B}{b^2} = \frac{\sin 2C}{c^2}$
- B-11.** A triangle is inscribed in a circle. The vertices of the triangle divide the circle into three arcs of length 3, 4 and 5 units. Then area of the triangle is equal to:
 (A) $\frac{9\sqrt{3}(1+\sqrt{3})}{\pi^2}$ (B) $\frac{9\sqrt{3}(\sqrt{3}-1)}{\pi^2}$ (C) $\frac{9\sqrt{3}(1+\sqrt{3})}{2\pi^2}$ (D) $\frac{9\sqrt{3}(\sqrt{3}-1)}{2\pi^2}$
- B-12.** In a $\triangle ABC$, $a = 1$ and the perimeter is six times the arithmetic mean of the sines of the angles. Then measure of $\angle A$ is
 (A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{4}$
- B-13*.** Three equal circles of radius unity touches one another. Radius of the circle touching all the three circles is :
 (A) $\frac{2-\sqrt{3}}{\sqrt{3}}$ (B) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$ (C) $\frac{2+\sqrt{3}}{\sqrt{3}}$ (D) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}}$
- B-14.** Triangle ABC is isosceles with $AB = AC$ and $BC = 65$ cm. P is a point on BC such that the perpendicular distances from P to AB and AC are 24 cm and 36 cm, respectively. The area of triangle ABC (in sq. cm) is
 (A) 1254 (B) 1950 (C) 2535 (D) 5070

Section (C) Inradius and Exradius

- C-1.** In a $\triangle ABC$, the value of $\frac{a\cos A + b\cos B + c\cos C}{a+b+c}$ is equal to:
 (A) $\frac{r}{R}$ (B) $\frac{R}{2r}$ (C) $\frac{R}{r}$ (D) $\frac{2r}{R}$
- C-2.** In a triangle ABC , if $a : b : c = 3 : 7 : 8$, then $R : r$ is equal to
 (A) 2 : 7 (B) 7 : 2 (C) 3 : 7 (D) 7 : 3
- C-3*.** If $r_1 = 2r_2 = 3r_3$, then
 (A) $\frac{a}{b} = \frac{4}{5}$ (B) $\frac{a}{b} = \frac{5}{4}$ (C) $\frac{a}{c} = \frac{3}{5}$ (D) $\frac{a}{c} = \frac{5}{3}$



- C-4*.** In a $\triangle ABC$, following relations hold good. In which case(s) the triangle is a right angled triangle?
 (A) $r_2 + r_3 = r_1 - r$ (B) $a^2 + b^2 + c^2 = 8R^2$ (C) $r_1 = s$ (D) $2R = r_1 - r$
- C-5.** The perimeter of a triangle ABC right angled at C is 70, and the inradius is 6, then $|a - b|$ equals
 (A) 1 (B) 2 (C) 8 (D) 9
- C-6.** In a triangle ABC , if $\frac{a-b}{b-c} = \frac{s-a}{s-c}$, then r_1, r_2, r_3 are in:
 (A) A.P. (B) G.P. (C) H.P. (D) none of these
- C-7.** If the incircle of the $\triangle ABC$ touches its sides at L, M and N as shown in the figure and if x, y, z be the circumradii of the triangles MIN, NIL and LIM respectively, where I is the incentre, then the product xyz is equal to :



- (A) Rr^2 (B) rR^2 (C) $\frac{1}{2}Rr^2$ (D) $\frac{1}{2}rR^2$
- C-8.** If in a $\triangle ABC$, $\frac{r}{r_1} = \frac{1}{2}$, then the value of $\tan \frac{A}{2} \left(\tan \frac{B}{2} + \tan \frac{C}{2} \right)$ is equal to :
 (A) 2 (B) $\frac{1}{2}$ (C) 1 (D) 3
- C-9.** If in a $\triangle ABC$, $\angle A = \frac{\pi}{2}$, then $\tan \frac{C}{2}$ is equal to
 (A) $\frac{a-c}{2b}$ (B) $\frac{a-b}{2c}$ (C) $\frac{a-c}{b}$ (D) $\frac{a-b}{c}$
- C-10.** In any $\triangle ABC$, $\frac{(r_1 + r_2)(r_2 + r_3)(r_3 + r_1)}{R^2}$ is always equal to
 (A) 8 (B) 27 (C) 16 (D) 4
- C-11*.** In a triangle ABC , right angled at B , then
 (A) $r = \frac{AB + BC - AC}{2}$ (B) $r = \frac{AB + AC - BC}{2}$
 (C) $r = \frac{AB + BC + AC}{2}$ (D) $R = \frac{s-r}{2}$
- C-12*.** With usual notations, in a $\triangle ABC$ the value of $\Pi (r_1 - r)$ can be simplified as:
 (A) $abc \Pi \tan \frac{A}{2}$ (B) $4rR^2$ (C) $\frac{(abc)^2}{R(a+b+c)^2}$ (D) $4Rr^2$

- C-13. STATEMENT-1 :** In a triangle ABC , the harmonic mean of the three exradii is three times the inradius.
STATEMENT-2 : In any triangle ABC , $r_1 + r_2 + r_3 = 4R$.
- (A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
 (B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
 (C) STATEMENT-1 is true, STATEMENT-2 is false
 (D) STATEMENT-1 is false, STATEMENT-2 is true
 (E) Both STATEMENTS are false



Section (D) Miscellaneous

- D-1.** If in a triangle ABC, the line joining the circumcentre and incentre is parallel to BC, then $\cos B + \cos C$ is equal to :
 (A) 0 (B) 1 (C) 2 (D) $\frac{1}{2}$
- D-2.** In a $\triangle ABC$, if $AB = 5$ cm, $BC = 13$ cm and $CA = 12$ cm, then the distance of vertex 'A' from the side BC is (in cm)
 (A) $\frac{25}{13}$ (B) $\frac{60}{13}$ (C) $\frac{65}{12}$ (D) $\frac{144}{13}$
- D-3.** If AD, BE and CF are the medians of a $\triangle ABC$, then $(AD^2 + BE^2 + CF^2) : (BC^2 + CA^2 + AB^2)$ is equal to
 (A) 4 : 3 (B) 3 : 2 (C) 3 : 4 (D) 2 : 3
- D-4.** In a triangle ABC, with usual notations the length of the bisector of internal angle A is :
 (A) $\frac{2bc \cos \frac{A}{2}}{b+c}$ (B) $\frac{2bc \sin \frac{A}{2}}{b+c}$ (C) $\frac{abc \operatorname{cosec} \frac{A}{2}}{2R(b+c)}$ (D) $\frac{2\Delta}{b+c} \cdot \operatorname{cosec} \frac{A}{2}$
- D-5.** Let f, g, h be the lengths of the perpendiculars from the circumcentre of the $\triangle ABC$ on the sides BC, CA and AB respectively. If $\frac{a}{f} + \frac{b}{g} + \frac{c}{h} = \lambda \frac{abc}{fgh}$, then the value of ' λ ' is:
 (A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) 1 (D) 2
- D-6.** In an acute angled triangle ABC, AP is the altitude. Circle drawn with AP as its diameter cuts the sides AB and AC at D and E respectively, then length DE is equal to
 (A) $\frac{\Delta}{2R}$ (B) $\frac{\Delta}{3R}$ (C) $\frac{\Delta}{4R}$ (D) $\frac{\Delta}{R}$
- D-7.** AA_1 , BB_1 and CC_1 are the medians of triangle ABC whose centroid is G. If points A, C_1 , G and B_1 are concyclic, then
 (A) $2b^2 = a^2 + c^2$ (B) $2c^2 = a^2 + b^2$ (C) $2a^2 = b^2 + c^2$ (D) $3a^2 = b^2 + c^2$
- D-8.** If ' ℓ ' is the length of median from the vertex A to the side BC of a $\triangle ABC$, then
 (A) $4\ell^2 = b^2 + 4ac \cos B$ (B) $4\ell^2 = a^2 + 4bc \cos A$
 (C) $4\ell^2 = c^2 + 4ab \cos C$ (D) $4\ell^2 = b^2 + 2c^2 - 2a^2$
- D-9.** The product of the distances of the incentre from the angular points of a $\triangle ABC$ is:
 (A) $4R^2r$ (B) $4Rr^2$ (C) $\frac{(abc)R}{s}$ (D) $\frac{(abc)r}{s}$
- D-10.** In a triangle ABC, $B = 60^\circ$ and $C = 45^\circ$. Let D divides BC internally in the ratio 1 : 3, then value of $\frac{\sin \angle BAD}{\sin \angle CAD}$ is
 (A) $\sqrt{\frac{2}{3}}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{6}}$ (D) $\frac{1}{3}$
- D-11.** In a triangle ABC, points D and E are taken on side BC such that $BD = DE = EC$. If angle $ADE = \text{angle } AED = \theta$, then:
 (A) $\tan \theta = 3 \tan B$ (B) $3 \tan \theta = \tan C$
 (C) $\frac{6 \tan \theta}{\tan^2 \theta - 9} = \tan A$ (D) angle B = angle C



D-12. STATEMENT-1 : If R be the circumradius of a $\triangle ABC$, then circumradius of its excentral $\triangle I_1 I_2 I_3$ is $2R$.

STATEMENT-2 : If circumradius of a triangle be R , then circumradius of its pedal triangle is $\frac{R}{2}$.

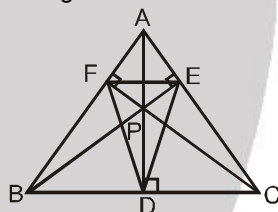
- (A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
 (B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
 (C) STATEMENT-1 is true, STATEMENT-2 is false
 (D) STATEMENT-1 is false, STATEMENT-2 is true
 (E) Both STATEMENTS are false

PART-II (COMPREHENSION)

Comprehension # 1 (Q. No. 1 to 4)

The triangle DEF which is formed by joining the feet of the altitudes of triangle ABC is called the Pedal Triangle.

Answer The Following Questions :



1. Angle of triangle DEF are
 (A) $\pi - 2A, \pi - 2B$ and $\pi - 2C$
 (B) $\pi + 2A, \pi + 2B$ and $\pi + 2C$
 (C) $\pi - A, \pi - B$ and $\pi - C$
 (D) $2\pi - A, 2\pi - B$ and $2\pi - C$
- 2*. Sides of triangle DEF are
 (A) $b \cos A, a \cos B, c \cos C$
 (B) $a \cos A, b \cos B, c \cos C$
 (C) $R \sin 2A, R \sin 2B, R \sin 2C$
 (D) $a \cot A, b \cot B, c \cot C$
3. Circumradii of the triangle PBC, PCA and PAB are respectively
 (A) R, R, R
 (B) $2R, 2R, 2R$
 (C) $R/2, R/2, R/2$
 (D) $3R, 3R, 3R$
- 4*. Which of the following is/are correct
 (A) $\frac{\text{Perimeter of } \triangle DEF}{\text{Perimeter of } \triangle ABC} = \frac{r}{R}$
 (B) Area of $\triangle DEF = 2 \Delta \cos A \cos B \cos C$
 (C) Area of $\triangle AEF = \Delta \cos^2 A$
 (D) Circum-radius of $\triangle DEF =$

Comprehension # 2 (Q. 5 to 8)

The triangle formed by joining the three excentres I_1, I_2 and I_3 of $\triangle ABC$ is called the excentral or excentric triangle and in this case internal angle bisector of triangle ABC are the altitudes of triangles $I_1 I_2 I_3$

5. Incentre I of $\triangle ABC$ is the of the excentral $\triangle I_1 I_2 I_3$.
 (A) Circumcentre
 (B) Orthocentre
 (C) Centroid
 (D) None of these
6. Angles of the $\triangle I_1 I_2 I_3$ are
 (A) $\frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2}$ and $\frac{\pi}{2} - \frac{C}{2}$
 (B) $\frac{\pi}{2} + \frac{A}{2}, \frac{\pi}{2} + \frac{B}{2}$ and $\frac{\pi}{2} + \frac{C}{2}$
 (C) $\frac{\pi}{2} - A, \frac{\pi}{2} - B$ and $\frac{\pi}{2} - C$
 (D) None of these



7. Sides of the $\Delta I_1 I_2 I_3$ are

- (A) $R \cos \frac{A}{2}, R \cos \frac{B}{2}, R \cos \frac{C}{2}$ (B) $4R \cos \frac{A}{2}, 4R \cos \frac{B}{2}, 4R \cos \frac{C}{2}$
 (C) $2R \cos \frac{A}{2}, 2R \cos \frac{B}{2}, 2R \cos \frac{C}{2}$ (D) None of these

8. Value of $I_1^2 + I_2^2 + I_3^2 = I_2^2 + I_3^2 + I_1^2 = I_3^2 + I_1^2 + I_2^2 =$

- (A) $4R^2$ (B) $16R^2$ (C) $32R^2$ (D) $64R^2$

PART-III (MATCH THE COLUMN)

1. Match the column

Column-I

Column-II

- (A) In a ΔABC , $2B = A + C$ and $b^2 = ac$. (p) 8
 Then the value of $\frac{a^2(a+b+c)}{3abc}$ is equal to
 (B) In any right angled triangle ABC, the value of $\frac{a^2 + b^2 + c^2}{R^2}$ (q) 1
 is always equal to (where R is the circumradius of ΔABC)
 (C) In a ΔABC if $a = 2$, $bc = 9$, then the value of $2R\Delta$ is equal to (r) 5
 (D) In a ΔABC , $a = 5$, $b = 3$ and $c = 7$, then the value of (s) 9
 $3 \cos C + 7 \cos B$ is equal to

2. Match the column

Column - I

Column - II

- (A) In a ΔABC , $a = 4$, $b = 3$ and the medians AA_1 and BB_1 are (p) 27
 mutually perpendicular, then square of area of the ΔABC
 is equal to
 (B) In any ΔABC , minimum value of $\frac{r_1 r_2 r_3}{r^3}$ is equal to (q) 7
 (C) In a ΔABC , $a = 5$, $b = 4$ and $\tan \frac{C}{2} = \sqrt{\frac{7}{9}}$, then side 'c' (r) 6
 is equal to
 (D) In a ΔABC , $2a^2 + 4b^2 + c^2 = 4ab + 2ac$, then value of $(8 \cos B)$ (s) 11
 is equal to

Exercise-3

Marked Questions may have for Revision Questions.

* Marked Questions may have more than one correct option.

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1. If the angle A, B and C of a triangle are in arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c} \sin 2C + \frac{c}{a} \sin 2A$ is

[IIT-JEE 2010, Paper-1, (3, -1), 84]

- (A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$



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ADVSOT - 18



2. Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are)
[IIT-JEE 2010, Paper-1, (3, 0), 84]
(A) $-(2 + \sqrt{3})$ (B) $1 + \sqrt{3}$ (C) $2 + \sqrt{3}$ (D) $4\sqrt{3}$
3. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. Suppose $a = 6$, $b = 10$ and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to
[IIT-JEE 2010, Paper-2, (3, 0), 79]
4. Let PQR be a triangle of area Δ with $a = 2$, $b = \frac{7}{2}$ and $c = \frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals
[IIT-JEE 2012, Paper-2, (3, -1), 66]
(A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$ (C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$
5. In a triangle PQR, P is the largest angle and $\cos P = \frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are)
[JEE (Advanced) 2013, Paper-2, (3, -1)/60]
(A) 16 (B) 18 (C) 24 (D) 22
6. In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is
[JEE (Advanced) 2014, Paper-2, (3, -1)/60]
(A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$ (C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$
- 7*. In a triangle XYZ, let x, y, z be the lengths of sides opposite to the angles X, Y, Z, respectively, and $2s = x + y + z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then
[JEE (Advanced) 2016, Paper-1, (4, -2)/62]
(A) area of the triangle XYZ is $6\sqrt{6}$
(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$
(C) $\sin \frac{X}{2} \sin \frac{Y}{2} \sin \frac{Z}{2} = \frac{4}{35}$
(D) $\sin^2 \left(\frac{X+Y}{2} \right) = \frac{3}{5}$
- 8*. In a triangle PQR, let $\angle PQR = 30^\circ$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE?
[JEE(Advanced) 2018, Paper-1, (4, -2)/60]
(A) $\angle QPR = 45^\circ$
(B) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120^\circ$
(C) The radius of the incircle of the triangle PQR is $10\sqrt{3} - 15$
(D) The area of the circumcircle of the triangle PQR is 100π





9. In a non-right-angled triangle ΔPQR , Let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S , the perpendicular from P meets the side QR at E , and RS and PE intersect at O . If $p = \sqrt{3}$, $q = 1$, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ?

[JEE(Advanced) 2019, Paper-1, (4, -1)/62]

- (A) Length of $RS = \frac{\sqrt{7}}{2}$ (B) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$
 (C) Radius of incircle of $\Delta PQR = \frac{\sqrt{3}}{2} (2 - \sqrt{3})$ (D) Length of $OE = \frac{1}{6}$

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is [AIEEE - 2010 (4, -1), 144]

- (1) There is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$. (2) There is a regular polygon with $\frac{r}{R} = \frac{2}{3}$.
 (3) There is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$. (4) There is a regular polygon with $\frac{r}{R} = \frac{1}{2}$.

2. ABCD is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, then AB is equal to : [AIEEE - 2013, (4, -1), 120]

- (1) $\frac{(p^2 + q^2) \sin \theta}{p \cos \theta + q \sin \theta}$ (2) $\frac{p^2 + q^2 \cos \theta}{p \cos \theta + q \sin \theta}$ (3) $\frac{p^2 + q^2}{p^2 \cos \theta + q^2 \sin \theta}$ (4) $\frac{(p^2 + q^2) \sin \theta}{(p \cos \theta + q \sin \theta)^2}$

3. With the usual notation, in ΔABC , if $\angle A + \angle B = 120^\circ$, $a = \sqrt{3} + 1$ and $b = \sqrt{3} - 1$, then the ratio $\angle A : \angle B$, is: [JEE(Main) 2019, Online (10-01-19), P-2 (4, -1), 120]

- (1) 9 : 7 (2) 7 : 1 (3) 3 : 1 (4) 5 : 3

4. In a triangle, the sum of lengths of two sides is x and the product of the lengths of the same two sides is y . If $x^2 - c^2 = y$, where c is the length of the third side of the triangle, then the circumradius of the triangle is [JEE(Main) 2019, Online (11-01-19), P-1 (4, -1), 120]

- (1) $\frac{c}{\sqrt{3}}$ (2) $\frac{3}{2}y$ (3) $\frac{c}{3}$ (4) $\frac{y}{\sqrt{3}}$



Answers

EXERCISE - 1

Section (A) :

A-2. $x > 5$ A-3. 75° A-8. $\frac{P}{2\sqrt{R}}$ A-11. 30° A-13. 10°

Section (B) :

B-2. $\frac{2}{3} \cot \frac{B}{2}$ B-3. 9 sq. unit

Section (C)

C-4. 6, 8, 10 cm C-5. 8

Section (D)

D-2. 3 D-3. $\sqrt{2}$

EXERCISE - 2

PART - I

Section (A) :

A-1. (C) A-2. (AB) A-3. (C) A-4. (C) A-5. (D) A-6. (C) A-7. (D)
A-8. (B) A-9. (AB) A-10. (A) A-11. (A)

Section (B) :

B-1. (B) B-2. (A) B-3. (A) B-4. (C) B-5. (B) B-6. (AD) B-7. (B)
B-8. (ABC) B-9. (B) B-10. (AB) B-11. (A) B-12. (C) B-13. (AC) B-14. (C)

Section (C) :

C-1. (A) C-2. (B) C-3. (BD) C-4. (ABCD) C-5. (A) C-6. (A) C-7. (C)
C-8. (B) C-9. (D) C-10. (D) C-11*. (AD) C-12*. (ACD) C-13. (C)

Section (D)

D-1. (B) D-2. (B) D-3. (C) D-4. (ACD) D-5. (A) D-6. (D) D-7. (C)
D-8. (B) D-9. (BD) D-10. (C) D-11. (ACD) D-12. (A)

PART - II

1. (A) 2. (BC) 3. (A) 4. (ABCD) 5. (B) 6. (A) 7. (B)
8. (B)

PART - III

1. (A) $\rightarrow$ (q), (B) $\rightarrow$ (p), (C) $\rightarrow$ (s), (D) $\rightarrow$ (r)
2. (A) $\rightarrow$ (s), (B) $\rightarrow$ (p), (C) $\rightarrow$ (r), (D) $\rightarrow$ (q)

EXERCISE - 3

PART - I

1. (D) 2. (B) 3. 3 4. (C) 5. (BD) 6. (B)
7. (A,C,D) 8. (BCD) 9. (ACD)

PART - II

1. (2) 2. (1) 3. (2) 4. (1)



HLP Answers

SUBJECTIVE QUESTIONS

This questions paste Staright Line sheets

- In $\triangle ABC$, P is an interior point such that $\angle PAB = 10^\circ$, $\angle PBA = 20^\circ$, $\angle PCA = 30^\circ$, $\angle PAC = 40^\circ$ then prove that $\triangle ABC$ is isosceles
- In a triangle ABC, if $a \tan A + b \tan B = (a + b) \tan \left(\frac{A+B}{2} \right)$, prove that triangle is isosceles.
- In any triangle ABC, if $2\Delta a - b^2c = c^3$, (where Δ is the area of triangle), then prove that $\angle A$ is obtuse
- If in a triangle ABC, $\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$ prove that the triangle ABC is either isosceles or right angled.
- In a $\triangle ABC$, $\angle C = 60^\circ$ and $\angle A = 75^\circ$. If D is a point on AC such that the area of the $\triangle BAD$ is $\sqrt{3}$ times the area of the $\triangle BCD$, find the $\angle ABD$.
- In a $\triangle ABC$, if a, b and c are in A.P., prove that $\cos A \cot \frac{A}{2}$, $\cos B \cot \frac{B}{2}$, and $\cos C \cot \frac{C}{2}$ are in A.P.
- In a triangle ABC, prove that the area of the incircle is to the area of triangle itself is,
 $\pi : \cot \left(\frac{A}{2} \right) \cdot \cot \left(\frac{B}{2} \right) \cdot \cot \left(\frac{C}{2} \right)$.
- In $\triangle ABC$, prove that $a^2(s-a) + b^2(s-b) + c^2(s-c) = 4R\Delta \left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right)$
- In any $\triangle ABC$, prove that
 - $(r_3 + r_1)(r_3 + r_2) \sin C = 2r_3 \sqrt{r_2 r_3 + r_3 r_1 + r_1 r_2}$
 - $\frac{\tan \frac{A}{2}}{(a-b)(a-c)} + \frac{\tan \frac{B}{2}}{(b-a)(b-c)} + \frac{\tan \frac{C}{2}}{(c-a)(c-b)} = \frac{1}{\Delta}$
 - $(r+r_1) \tan \frac{B-C}{2} + (r+r_2) \tan \frac{C-A}{2} + (r+r_3) \tan \frac{A-B}{2} = 0$
 - $r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - a^2 - b^2 - c^2$.
- In an acute angled triangle ABC, $r + r_1 = r_2 + r_3$ and $\angle B > \frac{\pi}{3}$, then prove that $b + 3c < 3a < 3b + 3c$
- If the inradius in a right angled triangle with integer sides is r. Prove that
 - If $r = 4$, the greatest perimeter (in units) is 90
 - If $r = 5$, the greatest area (in sq. units) is 330
- If $\left(1 - \frac{r_1}{r_2} \right) \left(1 - \frac{r_1}{r_3} \right) = 2$, then prove that the triangle is right angled.





13. DEF is the triangle formed by joining the points of contact of the incircle with the sides of the triangle ABC; prove that
- its sides are $2r \cos \frac{A}{2}$, $2r \cos \frac{B}{2}$ and $2r \cos \frac{C}{2}$,
 - its angles are $\frac{\pi}{2} - \frac{A}{2}$, $\frac{\pi}{2} - \frac{B}{2}$ and $\frac{\pi}{2} - \frac{C}{2}$
and
 - its area is $\frac{2\Delta^3}{(abc)s}$, i.e. $\frac{1}{2} \frac{r}{R} \Delta$.
14. Three circles, whose radii are a , b and c , touch one another externally and the tangents at their points of contact meet in a point, prove that the distance of this point from either of their points of contact is $\left(\frac{abc}{a+b+c} \right)^{\frac{1}{2}}$.
15. OA and OB are the equal sides of an isosceles triangle lying in the first quadrant making angles θ and ϕ respectively with x-axis. Show that the gradient of the bisector of acute angle AOB is $\operatorname{cosec} \beta - \cot \beta$ where $\beta = \phi + \theta$. (Where O is origin)
16. The hypotenuse $BC = a$ of a right-angled triangle ABC is divided into n equal segments where n is odd. The segment containing the midpoint of BC subtends angle α at A. Also h is the altitude of the triangle through A. Prove that $\tan \alpha = \frac{4nh}{a(n^2 - 1)}$.

Answers

5. $\angle ABD = 30^\circ$





Exercise-1

Marked questions are recommended for Revision.

चिह्नित प्रश्न दोहराने योग्य प्रश्न है।

SUBJECTIVE QUESTIONS

विषयात्मक प्रश्न (SUBJECTIVE QUESTIONS)

Section (A) : Sine rule, Cosine rule, Napier's Analogy, Projection rule

खण्ड (A) : ज्या नियम, कोज्या नियम, स्पर्शज्या नियम या नेपियर एनालोजि, प्रक्षेप नियम

A-1. In a $\triangle ABC$, prove that :

त्रिभुज ABC में सिद्ध कीजिए कि :

(i) $a \sin(B - C) + b \sin(C - A) + c \sin(A - B) = 0$

(ii) $\frac{a^2 \sin(B - C)}{\sin A} + \frac{b^2 \sin(C - A)}{\sin B} + \frac{c^2 \sin(A - B)}{\sin C} = 0$

(iii) $2(bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$

(iv) $(a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2} = c^2$

(v) $b^2 \sin 2C + c^2 \sin 2B = 2bc \sin A$

(vi) $\frac{\sin B}{\sin C} = \frac{c - a \cos B}{b - a \cos C}$

Sol.

(i) L.H.S. = $a \sin(B - C) + b \sin(C - A) + c \sin(A - B)$
 $= k \sin A \sin(B - C) + k \sin B \sin(C - A) + k \sin C \sin(A - B)$
 $= k (\sin^2 B - \sin^2 C) + k (\sin^2 C - \sin^2 A) + k (\sin^2 A - \sin^2 B)$
 $= 0 = \text{R.H.S.}$

(ii) L.H.S. = $\frac{a^2 \sin(B - C)}{\sin A} + \frac{b^2 \sin(C - A)}{\sin B} + \frac{c^2 \sin(A - B)}{\sin C}$

first term = $\frac{a^2 \sin(B - C)}{\sin A} = \frac{k^2 \sin^2 A \sin(B - C)}{\sin A}$
 $= k^2 \sin(B + C) \sin(B - C)$
 $= k^2 (\sin^2 B - \sin^2 C)$

Similarly $\frac{b^2 \sin(C - A)}{\sin B} = k^2 (\sin^2 C - \sin^2 A)$

and $\frac{c^2 \sin(A - B)}{\sin C} = k^2 (\sin^2 A - \sin^2 B)$

$\therefore$ L.H.S. = $k^2 (\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B)$
 $= 0 = \text{R.H.S.}$

(iii) L.H.S. = $2bc \cos A + 2ca \cos B + 2ab \cos C$
 $= b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2$
 $= a^2 + b^2 + c^2$
 $= \text{R.H.S.}$

(iv) L.H.S. = $a^2 \left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2} \right) + b^2 \left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2} \right) - 2ab \left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2} \right)$
 $= a^2 + b^2 - 2ab \cos C$
 $= a^2 + b^2 - (a^2 + b^2 - c^2)$
 $= c^2 = \text{R.H.S.}$

(v) $\therefore$ L.H.S. = $b^2 \sin 2C + c^2 \sin 2B$
 $= 2b^2 \sin C \cos C + 2c^2 \sin B \cos B$
 $= 2k^2 \sin^2 B \cos C \sin C + 2k^2 \sin^2 C \sin B \cos B$ ($\because b = k \sin B, c = k \sin C$)
 $= 2k^2 \sin B \sin C [\sin B \cos C + \cos B \sin C]$
 $= 2(k \sin B)(k \sin C) \sin(B + C)$





$$= 2bc \sin A$$

$$(vi) \therefore R.H.S = \frac{c - a \cos B}{b - a \cos C} \quad \because c = a \cos B + b \cos A,$$

$$b = c \cos A + a \cos C$$

$$= \frac{b \cos A}{c \cos A} = \frac{b}{c}$$

$$= \frac{\sin B}{\sin C} = L.H.S.$$

Hindi. (i) बायां पक्ष (L.H.S.) = $a \sin (B - C) + b \sin (C - A) + c \sin (A - B)$
 $= k \sin A \sin (B - C) + k \sin B \sin (C - A) + k \sin C \sin (A - B)$
 $= k (\sin^2 B - \sin^2 C) + k (\sin^2 C - \sin^2 A) + k (\sin^2 A - \sin^2 B)$
 $= 0 = दायां पक्ष (R.H.S.)$

(ii) बायां पक्ष (L.H.S.) = $\frac{a^2 \sin(B - C)}{\sin A} + \frac{b^2 \sin(C - A)}{\sin B} + \frac{c^2 \sin(A - B)}{\sin C}$

प्रथम पद = $\frac{a^2 \sin(B - C)}{\sin A} = \frac{k^2 \sin^2 A \sin(B - C)}{\sin A}$
 $= k^2 \sin (B + C) \sin (B - C)$
 $= k^2 (\sin^2 B - \sin^2 C)$

इसी प्रकार, $\frac{b^2 \sin(C - A)}{\sin B} = k^2 (\sin^2 C - \sin^2 A)$

और $\frac{c^2 \sin(A - B)}{\sin C} = k^2 (\sin^2 A - \sin^2 B)$

$\therefore$ बायां पक्ष (L.H.S.) = $k^2 (\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B)$
 $= 0 = दायां पक्ष (R.H.S.)$

(iii) बायां पक्ष (L.H.S.) = $2bc \cos A + 2ca \cos B + 2ab \cos C$
 $= b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2$
 $= a^2 + b^2 + c^2$
 $= R.H.S. दायां पक्ष (R.H.S.)$

(iv) बायां पक्ष (L.H.S.) = $a^2 \left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2} \right) + b^2 \left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2} \right) - 2ab \left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2} \right)$
 $= a^2 + b^2 - 2ab \cos C$
 $= a^2 + b^2 - (a^2 + b^2 - c^2)$
 $= c^2 = दायां पक्ष (R.H.S.)$

(v) $\therefore$ बायां पक्ष (L.H.S.) = $b^2 \sin 2C + c^2 \sin 2B$
 $= 2b^2 \sin C \cos C + 2c^2 \sin B \cos B$
 $= 2k^2 \sin^2 B \cos C \sin C + 2k^2 \sin^2 C \sin B \cos B (\because b = k \sin B, c = k \sin C)$
 $= 2k^2 \sin B \sin C [\sin B \cos C + \cos B \sin C]$
 $= 2(k \sin B) (k \sin C) \sin (B + C)$
 $= 2bc \sin A$

(vi) $\therefore$ दायां पक्ष (R.H.S.) = $\frac{c - a \cos B}{b - a \cos C} \quad \because c = a \cos B + b \cos A,$
 $b = c \cos A + a \cos C$

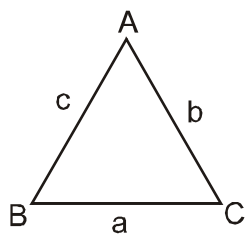
$$= \frac{b \cos A}{c \cos A} = \frac{b}{c}$$

$$= \frac{\sin B}{\sin C} = बायां पक्ष (L.H.S.)$$

A-2. Find the real value of x such that $x^2 + 2x$, $2x + 3$ and $x^2 + 3x + 8$ are lengths of the sides of a triangle.
 x के वह वास्तविक मान ज्ञात कीजिए जबकि $x^2 + 2x$, $2x + 3$ तथा $x^2 + 3x + 8$ एक त्रिभुज की भुजायें हो।

Ans. $x > 5$

Sol.



For $\triangle ABC$ $a + b > c$,
 $x^2 + 4x + 3 > x^2 + 3x + 8 \Rightarrow$

$x^2 + 5x + 11 > x^2 + 2x$

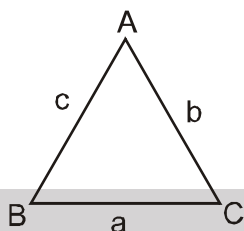
$2x^2 + 5x + 8 > 2x + 3$
 Common to all is $x > 5$.

$b + c > a$, $c + a > b$
 $x > 5$

$\Rightarrow x > \frac{-11}{3}$

$\Rightarrow 2x^2 + 3x + 5 > 0 \Rightarrow x \in \mathbb{R}$

Hindi.



$\triangle ABC$ के लिए $a + b > c$,
 $x^2 + 4x + 3 > x^2 + 3x + 8 \Rightarrow$

$x^2 + 5x + 11 > x^2 + 2x$

$2x^2 + 5x + 8 > 2x + 3$

$\Rightarrow 2x^2 + 3x + 5 > 0 \Rightarrow$

सभी के लिए उभयनिष्ठ $x > 5$.

$b + c > a$, $c + a > b$
 $x > 5$

$\Rightarrow x > \frac{-11}{3}$

$x \in \mathbb{R}$

A-3. The angles of a $\triangle ABC$ are in A.P. (order being A, B, C) and it is being given that $b : c = \sqrt{3} : \sqrt{2}$, then find $\angle A$.

त्रिभुज ABC के कोण A, B, C इसी क्रम में समान्तर श्रेढ़ी में है तथा $b : c = \sqrt{3} : \sqrt{2}$ है, तो $\angle A$ ज्ञात कीजिए।

Ans. 75°

Sol.

$\therefore 2B = A + C$,

$\Rightarrow B = 60^\circ$

from Sine-rule ज्या नियम के अनुसार

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{b}{c} = \frac{\sin B}{\sin C} = \frac{\sqrt{3}}{\sqrt{2}}$

$\therefore \sin C = \frac{1}{\sqrt{2}} \Rightarrow C = 45^\circ$

$\therefore A = 75^\circ$

A-4. If $\cos A + \cos B = 4 \sin^2\left(\frac{C}{2}\right)$, prove that sides a, c, b of the triangle ABC are in A.P.

यदि $\cos A + \cos B = 4 \sin^2\left(\frac{C}{2}\right)$ हो, तो सिद्ध कीजिए कि त्रिभुज ABC की भुजाएँ a, c, b समान्तर श्रेढ़ी में हैं।

Sol.

$\therefore \cos A + \cos B = 4 \sin^2 \frac{C}{2} \Rightarrow 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) = 4 \sin^2 \frac{C}{2}$

$\Rightarrow 2 \cos\left(\frac{A-B}{2}\right) = 4 \sin \frac{C}{2} \Rightarrow 2 \cos \frac{C}{2} \cos\left(\frac{A-B}{2}\right) = 4 \sin \frac{C}{2} \cos \frac{C}{2}$

$\Rightarrow 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) = 2 \sin C \Rightarrow \sin A + \sin B = 2 \sin C$





$$\Rightarrow a + b = 2c \quad \Rightarrow a, c, b \text{ are in A.P.} \quad a, c, b \text{ समान्तर श्रेणी में है।}$$

A-5. If in a $\triangle ABC$, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, then prove that a^2, b^2, c^2 are in A.P.

यदि त्रिभुज ABC में, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$ हो, तो सिद्ध कीजिए कि a^2, b^2, c^2 समान्तर श्रेणी में है।

Sol. $\therefore \frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)} \Rightarrow \sin(B+C) \sin(B-C) = \sin(A+B) \sin(A-B)$
 $\Rightarrow \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B \quad \Rightarrow 2 \sin^2 B = \sin^2 A + \sin^2 C$
 $\Rightarrow 2b^2 = a^2 + c^2 \quad \Rightarrow a^2, b^2, c^2 \text{ are in A.P.} \quad \Rightarrow a^2, b^2, c^2 \text{ समान्तर श्रेणी में होंगे।}$

A-6. In a triangle ABC , prove that for any angle θ , $b \cos(A-\theta) + a \cos(B+\theta) = c \cos \theta$.
 त्रिभुज ABC में किसी कोण θ के लिए सिद्ध कीजिए कि $b \cos(A-\theta) + a \cos(B+\theta) = c \cos \theta$.

Sol. $\therefore$ L.H.S. बायां पक्ष $= b(\cos A \cos \theta + \sin A \sin \theta) + a(\cos B \cos \theta - \sin B \sin \theta)$
 $= \cos \theta (b \cos A + a \cos B) + \sin \theta (b \sin A - a \sin B)$
 $= c \cos \theta + 0 \quad \{\because b \sin A - a \sin B = 0\}$
 $= c \cos \theta = \text{R.H.S. दायां पक्ष}$

A-7. With usual notations, if in a $\triangle ABC$, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, then prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$.

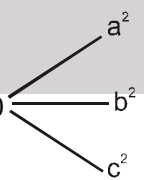
यदि त्रिभुज ABC में, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$ हो, तो सिद्ध कीजिए कि $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$.

Sol. $\therefore \frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} = k \quad \therefore \begin{cases} b+c=11k \\ c+a=12k \\ a+b=13k \end{cases} \Rightarrow a=7k, b=6k, c=5k$
 $\therefore \cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{36+25-49}{2 \times 6 \times 5} = \frac{1}{5}$
 $\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{25+49-36}{2 \times 5 \times 7} = \frac{19}{35}$
 $\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{49+36-25}{2 \times 7 \times 6} = \frac{5}{7} \quad \therefore \frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$

A-8. Let a, b and c be the sides of a $\triangle ABC$. If a^2, b^2 and c^2 are the roots of the equation $x^3 - Px^2 + Qx - R = 0$, where P, Q & R are constants, then find the value of $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$ in terms of P, Q and R .

माना a, b तथा c त्रिभुज ABC की भुजाएँ हैं। यदि a^2, b^2 एवं c^2 समीकरण $x^3 - Px^2 + Qx - R = 0$, जहाँ P, Q तथा R अचर हैं, के मूल हों तो $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$ का मान P, Q एवं R के पदों में ज्ञात कीजिए।

Ans. $\frac{P}{2\sqrt{R}}$

Sol. $\therefore x^3 - Px^2 + Qx - R = 0$ 
 $\therefore a^2 + b^2 + c^2 = P$
 $a^2b^2 + b^2c^2 + c^2a^2 = Q$
 $a^2b^2c^2 = R \Rightarrow abc = \sqrt{R}$
 $\therefore \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{1}{2abc} [a^2 + b^2 + c^2] = \frac{P}{2\sqrt{R}}$

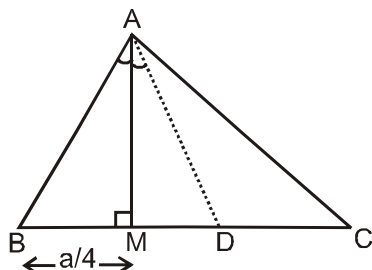




A-9. If in a triangle ABC, the altitude AM be the bisector of $\angle BAD$, where D is the mid point of side BC, then prove that $(b^2 - c^2) = a^2/2$.

यदि त्रिभुज ABC में शीर्षलम्ब AM कोण BAD का अर्धक हो, जहाँ D भुजा BC का मध्य बिन्दु है, तो सिद्ध कीजिए कि $(b^2 - c^2) = a^2/2$.

Sol.



$$\therefore \frac{a}{4} = c \cos B$$

$$\frac{a}{4} = c \left(\frac{a^2 + c^2 - b^2}{2ac} \right)$$

$$\therefore \frac{a^2}{2} = a^2 + c^2 - b^2$$

$$\therefore b^2 - c^2 = \frac{a^2}{2}$$

A-10. If in a triangle ABC, $\angle C = 60^\circ$, then prove that $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

यदि ABC में $\angle C = 60^\circ$ तब सिद्ध कीजिए कि $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

Sol. By the cosine formula, we have

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\text{or } c^2 = a^2 + b^2 - 2ab \cos 60^\circ = a^2 + b^2 - ab \quad \dots (i)$$

$$\text{Now, } \frac{1}{a+c} + \frac{1}{b+c} - \frac{3}{a+b+c} = \left[\frac{(b+c)(a+b+c) + (a+c)(a+b+c) - 3(a+c)(b+c)}{(a+b)(b+c)(a+b+c)} \right]$$

$$= \frac{(a^2 + b^2 - ab) - c^2}{(a+b)(b+c)(a+b+c)} = 0 \quad [\text{from eq. (i)}]$$

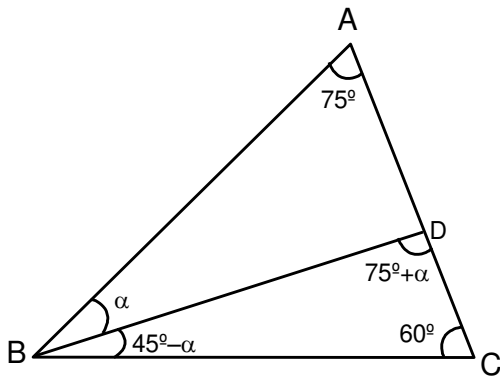
$$\text{or } \frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$$

A-11. In a triangle ABC, $\angle C = 60^\circ$ and $\angle A = 75^\circ$. If D is a point on AC such that the area of the $\triangle ABD$ is $\sqrt{3}$ times the area of the $\triangle BCD$, find the $\angle ABD$.

त्रिभुज ABC में $\angle C = 60^\circ$ तथा $\angle A = 75^\circ$. यदि D, AC पर बिन्दु इस प्रकार है कि $\triangle ABD$ का क्षेत्रफल, $\triangle BCD$ के क्षेत्रफल का $\sqrt{3}$ गुना है तब $\angle ABD$ ज्ञात कीजिए।

Ans. 30°

Sol. Let h be the length of perpendicular from B on AC



Given that $\frac{\Delta BAD}{\Delta BCD} = \sqrt{3}$

$$\Rightarrow \frac{\frac{1}{2}h \cdot AD}{\frac{1}{2}h \cdot DC} = \frac{AD}{DC} = \sqrt{3} \quad \dots(i)$$

In ΔBAD , taking $\angle ABD = \alpha$, we have

$$\frac{AD}{\sin \alpha} = \frac{BD}{\sin 75^\circ} \quad \dots(ii)$$

And in ΔBCD , we have $\frac{CD}{\sin(45^\circ - \alpha)} = \frac{BD}{\sin 60^\circ} \quad \dots(iii)$

$\therefore$ From (2) and (3), we get

$$\frac{AD \sin(45^\circ - \alpha)}{CD \sin \alpha} = \frac{\sin 60^\circ}{\sin 75^\circ}$$

$$\Rightarrow \sqrt{3} \left(\frac{\sqrt{3} + 1}{2\sqrt{2}} \right) \left(\frac{\cos \alpha - \sin \alpha}{\sqrt{2}} \right) = \frac{\sqrt{3}}{2} \sin \alpha$$

$$\Rightarrow \sqrt{\sqrt{3} + 1} \cos \alpha = (3 + \sqrt{3}) \sin \alpha$$

$$\Rightarrow \tan \alpha = 1/\sqrt{3}$$

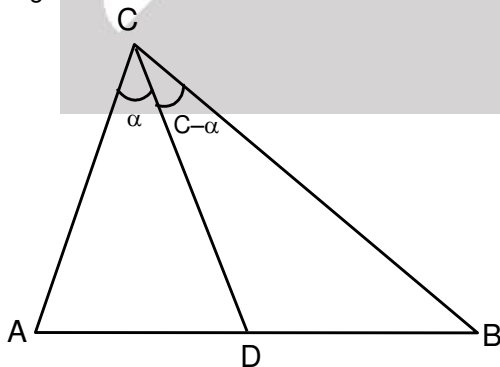
$$\Rightarrow \alpha = \pi/6 = 30^\circ$$

Hence $\angle ABD = 30^\circ$

A-12. In a scalene triangle ABC, D is a point on the side AB such that $CD^2 = AD \cdot DB$, if $\sin A \cdot \sin B = \sin^2 \frac{C}{2}$ then prove that CD is internal bisector of $\angle C$.

एक विषमबाहु त्रिभुज ABC में D भुजा AB पर एक बिन्दु D इस प्रकार है $CD^2 = AD \cdot DB$, यदि $\sin A \cdot \sin B = \sin^2 \frac{C}{2}$ तब सिद्ध कीजिए कि CD, कोण C का आन्तरिक अर्द्धक है।

Sol. Let $\angle ACD = \alpha \Rightarrow \angle DCB = (C - \alpha)$
Figure





Applying the sine rule in $\triangle ACD$ and in $\triangle DCB$ respectively, we get

$$\Rightarrow \frac{AD}{\sin \alpha} = \frac{CD}{\sin A} \text{ and } \frac{BD}{\sin(C-\alpha)} = \frac{CD}{\sin B} \Rightarrow \frac{AD \cdot BD}{\sin \alpha \sin(C-\alpha)} = \frac{CD^2}{\sin A \cdot \sin B}$$

$$\Rightarrow \frac{1}{2} [\cos(2\alpha - C) - \cos C] = \sin^2 \frac{C}{2}$$

$$\Rightarrow \frac{1}{2} [\cos(2\alpha - C) - 1 + 2\sin^2 \frac{C}{2}] = \sin^2 \frac{C}{2}$$

$$\Rightarrow \cos(2\alpha - C) = 1$$

$$\Rightarrow \alpha = \frac{C}{2}$$

Thus, CD is the internal angle bisector of angle C.

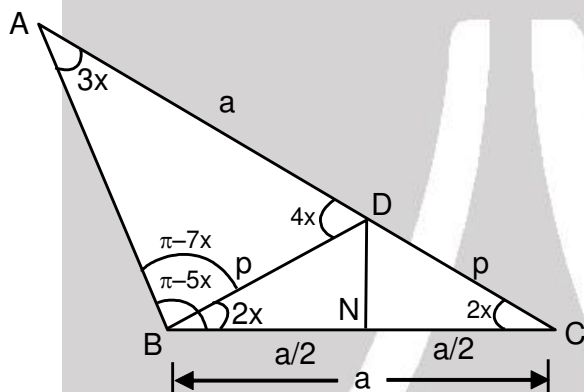
A-13. In triangle ABC, D is on AC such that $AD = BC$, $BD = DC$, $\angle DBC = 2x$, and $\angle BAD = 3x$, all angles are in degrees, then find the value of x.

$\triangle ABC$ में D, AC पर इस प्रकार है कि $AD = BC$, $BD = DC$, $\angle DBC = 2x$, और $\angle BAD = 3x$, सभी कोण, डिग्री में है तब x का मान ज्ञात कीजिए।

Ans. 10°

Sol. In $\triangle ABC$,

$$\frac{AC}{\sin 5x} = \frac{BC}{\sin 3x} \Rightarrow \frac{a+p}{\sin 5x} = \frac{a}{\sin 3x}$$



$$\text{In } \triangle BDN, \cos 2x = \frac{a}{2p}$$

$$\text{or } a = 2p \cos 2x$$

$$\text{From eq. (i), } \frac{2p \cos 2x + p}{\sin 5x} = \frac{2p \cos 2x}{\sin 3x}$$

$$\text{or } 2 \sin 3x \cos 2x + \sin 3x = 2 \sin 5x \cos 2x$$

$$\text{or } \sin 5x + \sin x + \sin 3x = \sin 7x + \sin 3x$$

$$\text{or } \sin 7x - \sin 5x = \sin x$$

$$\text{or } 2 \cos 6x \sin x = \sin x$$

$$\text{or } \cos 6x = \frac{1}{2} \Rightarrow x = 10^\circ$$

Section (B) Trigonometric ratios of Half Angles, Area of triangle and circumradius

खण्ड (B) अर्द्धकोण, त्रिभुज का क्षेत्रफल और परित्रिज्या में त्रिकोणमितिय सम्बन्ध

B-1. In a $\triangle ABC$, prove that
त्रिभुज ABC में सिद्ध कीजिए कि –



$$(i) \quad 2 \left[a \sin^2 \frac{C}{2} + c \sin^2 \frac{A}{2} \right] = c + a - b.$$

$$(ii) \quad \frac{\cos^2 \frac{A}{2}}{a} + \frac{\cos^2 \frac{B}{2}}{b} + \frac{\cos^2 \frac{C}{2}}{c} = \frac{s^2}{abc}$$

$$(iii) \quad 4 \left(bc \cos^2 \frac{A}{2} + ca \cos^2 \frac{B}{2} + ab \cos^2 \frac{C}{2} \right) = (a + b + c)^2$$

$$(iv) \quad (b - c) \cot \frac{A}{2} + (c - a) \cot \frac{B}{2} + (a - b) \cot \frac{C}{2} = 0$$

$$(v) \quad 4\Delta (\cot A + \cot B + \cot C) = a^2 + b^2 + c^2$$

$$(vi) \quad \left(\frac{2abc}{a+b+c} \right) \cdot \cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2} = \Delta$$

Sol. (i) L.H.S. (बायां पक्ष) = $2a \sin^2 \frac{C}{2} + 2c \sin^2 \frac{A}{2}$
 $= a(1 - \cos C) + c(1 - \cos A)$
 $= a + c - (a \cos C + c \cos A)$
 $= a + c - b$
 $= \text{R.H.S. (दायां पक्ष)}$

$$(ii) \quad \therefore \text{L.H.S. (बायां पक्ष)} = \frac{\cos^2 \frac{A}{2}}{a} + \frac{\cos^2 \frac{B}{2}}{b} + \frac{\cos^2 \frac{C}{2}}{c}$$

$$= \frac{1}{a} \cdot \frac{s(s-a)}{bc} + \frac{1}{b} \cdot \frac{s(s-b)}{ca} + \frac{1}{c} \cdot \frac{s(s-c)}{ab} = \frac{s(3s - (a+b+c))}{abc} = \frac{s^2}{abc}.$$

$$(iii) \quad \text{L.H.S. (बायां पक्ष)} = 2bc(1 + \cos A) + 2ca(1 + \cos B) + 2ab(1 + \cos C)$$

$$= 2bc + 2ca + 2ab + 2bc \cos A + 2ca \cos B + 2ab \cos C$$

$$= 2 \sum ab + a^2 + b^2 + c^2 = (a + b + c)^2 = \text{R.H.S. (दायां पक्ष)}$$

$$(iv) \quad \therefore \text{L.H.S. (बायां पक्ष)} = (b - c) \cot \frac{A}{2} + (c - a) \cot \frac{B}{2} + (a - b) \cot \frac{C}{2}$$

$$\therefore (b - c) \cot \frac{A}{2} = k(\sin B - \sin C) \frac{\cos \frac{A}{2}}{\sin \frac{A}{2}} = 2k \cos \left(\frac{B+C}{2} \right) \sin \left(\frac{B-C}{2} \right) \frac{\cos \frac{A}{2}}{\sin \frac{A}{2}}$$

$$= 2k \sin \left(\frac{B+C}{2} \right) \sin \left(\frac{B-C}{2} \right) = k [\cos C - \cos B]$$

similarly (इसी प्रकार) $(c - a) \cot \frac{B}{2} = k[\cos A - \cos C]$

and (और) $(a - b) \cot \frac{C}{2} = k[\cos B - \cos A]$

$$\therefore \text{L.H.S. (बायां पक्ष)} = k[\cos C - \cos B + \cos A - \cos C + \cos B - \cos A]$$

$$= 0$$

$$= \text{R.H.S. (दायां पक्ष)}$$

$$(v) \quad \text{L.H.S. बायां पक्ष} = 4\Delta (\cot A + \cot B + \cot C)$$

$$= 4\Delta \left(\frac{\cos A}{\sin A} + \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C} \right) \quad \left\{ \because \Delta = \frac{1}{2} bc \sin A \right\}$$

$$= 2bc \cos A + 2ca \cos B + 2ab \cos C$$

$$= a^2 + b^2 + c^2 = \text{R.H.S. दायां पक्ष}$$

$$(vi) \quad \text{L.H.S. बायां पक्ष} = \frac{2abc}{a+b+c} \cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}$$



$$= \frac{2}{2s} \frac{abc}{bc} \sqrt{\frac{s(s-a)}{bc} \cdot \frac{s(s-b)}{ca} \cdot \frac{s(s-c)}{ab}} = \sqrt{s(s-a)(s-b)(s-c)} = \Delta = \text{R.H.S. दायां पक्ष .}$$

B-2. If the sides a, b, c of a triangle are in A.P., then find the value of $\tan \frac{A}{2} + \tan \frac{C}{2}$ in terms of $\cot (B/2)$.

यदि किसी त्रिभुज की भुजाएँ a, b, c समान्तर श्रेढ़ी में हों, तो $\tan \frac{A}{2} + \tan \frac{C}{2}$ का मान $\cot (B/2)$ के पदों में ज्ञात कीजिए।

Ans. $\frac{2}{3} \cot \frac{B}{2}$

Sol. $\because 2b = a + c \quad \dots(i)$

$$\begin{aligned} \therefore \tan \frac{A}{2} + \tan \frac{C}{2} &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \\ &= \sqrt{\frac{s-b}{s}} \left[\frac{s-c+s-a}{\sqrt{(s-a)(s-c)}} \right] = \frac{b}{s} \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} \\ &= \frac{2b}{2s} \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = \frac{2b}{3b} \cot \frac{B}{2} = \frac{2}{3} \cot \frac{B}{2}. \end{aligned}$$

B-3. If in a ΔABC , $a = 6, b = 3$ and $\cos(A - B) = 4/5$, then find its area.

यदि त्रिभुज ABC में, $a = 6, b = 3$ एवं $\cos(A - B) = 4/5$ हो, तो इसका क्षेत्रफल ज्ञात कीजिए।

Ans. 9 sq. unit

Sol. $\because a = 6, b = 3$ and तथा $\cos(A - B) = 4/5$

$$\therefore \tan \left(\frac{A-B}{2} \right) = \frac{a-b}{a+b} \cot \frac{C}{2} \quad \dots (i)$$

$$\therefore \tan^2 \left(\frac{A-B}{2} \right) = \frac{1 - \cos(A-B)}{1 + \cos(A-B)} = \frac{1}{9}$$

$$\therefore \tan \left(\frac{A-B}{2} \right) = \frac{1}{3}$$

$$\therefore \text{from (i) से, we get } \frac{1}{3} = \frac{1}{3} \cot \frac{C}{2} \Rightarrow C = 90^\circ$$

$$\therefore \text{Area क्षेत्रफल} = \frac{1}{2} ab = 9 \text{ sq. unit वर्ग इकाई}$$

B-4. If in a triangle ABC , $\angle A = 30^\circ$ and the area of triangle is $\frac{\sqrt{3} a^2}{4}$, then prove that either $B = 4C$ or $C = 4B$.

यदि त्रिभुज ABC में, $\angle A = 30^\circ$ तथा त्रिभुज का क्षेत्रफल $\frac{\sqrt{3} a^2}{4}$ हो, तो सिद्ध कीजिए कि या तो $B = 4C$ या $C = 4B$.

Sol. $\because \angle A = 30^\circ$ and तथा $\Delta = \frac{\sqrt{3} a^2}{4}$

$$\therefore \frac{1}{2} bc \sin A = \frac{\sqrt{3}}{4} a^2 \Rightarrow \frac{1}{2} bc \sin 30^\circ = \frac{\sqrt{3}}{4} a^2 \Rightarrow bc = \sqrt{3} a^2$$

$$\Rightarrow \sin B \sin C = \sqrt{3} \sin^2 A \Rightarrow \sin B \sin C = \frac{\sqrt{3}}{4} \quad \text{as जैसा कि } \sin A = \frac{1}{2}$$

$$\Rightarrow \cos(B - C) - \cos(B + C) = \frac{\sqrt{3}}{2} \Rightarrow \cos(B - C) + \cos A = \frac{\sqrt{3}}{2}$$



$$\Rightarrow \cos(B - C) = 0 \quad (\text{as जैसा कि } \angle A = 30^\circ \Rightarrow \cos A = \frac{\sqrt{3}}{2})$$

$$\Rightarrow B - C = 90^\circ \quad \text{or या} \quad B - C = -90^\circ$$

But लेकिन $B + C = 150^\circ$ as $A = 30^\circ$ (क्योंकि दिया है $A = 30^\circ$)

case (i) : if यदि $B - C = 90^\circ$
and तथा $B + C = 150^\circ \Rightarrow B = 120^\circ$ and तथा $C = 30^\circ \Rightarrow B = 4C$

case (ii) : if यदि $B - C = -90^\circ$
and तथा $B + C = 150^\circ \Rightarrow B = 30^\circ$ and तथा $C = 120^\circ \therefore C = 4B.$

Section (C) Inradius and Exradius

खण्ड(C) अन्तत्रिज्या परित्रिज्या

C-1. In any $\triangle ABC$, prove that
त्रिभुज ABC में सिद्ध कीजिए कि -

$$(i) \quad Rr(\sin A + \sin B + \sin C) = \Delta \quad (ii) \quad a \cos B \cos C + b \cos C \cos A + c \cos A \cos B = \frac{\Delta}{R}$$

$$(iii) \quad \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr} \quad (iv) \quad \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = 2 + \frac{r}{2R}$$

$$(v) \quad a \cot A + b \cot B + c \cot C = 2(R + r)$$

Sol.

$$(i) \therefore \text{L.H.S. (बायां पक्ष)} = Rr(\sin A + \sin B + \sin C)$$

$$= Rr \left(\frac{a+b+c}{2R} \right) \quad \because r = \frac{\Delta}{s}$$

$$= \frac{r(2s)}{2} = rs = \Delta$$

$$(ii) \therefore \text{L.H.S. (बायां पक्ष)} = a \cos B \cos C + \cos A(b \cos C + c \cos B)$$

$$\begin{aligned} &= a[\cos B \cos C + \cos A] \\ &= a[\cos B \cos C - \cos(B + C)] \\ &= a \sin B \sin C \\ &= a \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{abc}{4R^2} = \frac{4R\Delta}{4R^2} = \frac{\Delta}{R} \end{aligned}$$

$$(iii) \therefore \text{L.H.S. (बायां पक्ष)} = \frac{c+a+b}{abc} = \frac{2s}{4R\Delta} = \frac{1}{2R \frac{\Delta}{s}} = \frac{1}{2Rr}$$

$$(iv) \quad \text{L.H.S. बायां पक्ष} = \frac{1}{2} (1 + \cos A + 1 + \cos B + 1 + \cos C)$$

$$= \frac{1}{2} \left(3 + 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) = 2 + \frac{1}{2} \frac{r}{R} = 2 + \frac{r}{2R}$$

$$(v) \quad \text{L.H.S. बायां पक्ष} = a \frac{\cos A}{\sin A} + b \frac{\cos B}{\sin B} + c \frac{\cos C}{\sin C}$$

$$= 2R \left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) = 2R + 2r = 2(R + r)$$

C-2. In any $\triangle ABC$, prove that
किसी त्रिभुज ABC में सिद्ध कीजिए कि :-

$$(i) \quad r_1 \cdot r_2 \cdot r_3 = \Delta^2$$

$$(ii) \quad r_1 + r_2 - r_3 + r = 4R \cos C.$$

$$(iii) \quad \frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2 + b^2 + c^2}{\Delta^2}$$





$$(iv) \left(\frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right)^2 = \frac{4}{r} \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right)$$

$$(v) \frac{bc - r_2 r_3}{r_1} = \frac{ca - r_3 r_1}{r_2} = \frac{ab - r_1 r_2}{r_3} = r$$

Sol. (i) $r \cdot r_1 \cdot r_2 \cdot r_3 = \frac{\Delta^4}{s(s-a)(s-b)(s-c)} = D^2$

$$(ii) \quad r_1 + r_2 + r_3 + r = 4R \cos C \quad \therefore \quad \text{L.H.S.} = \frac{\Delta}{s-a} + \frac{\Delta}{s-b} - \frac{\Delta}{s-c} + \frac{\Delta}{s}$$

$$= \Delta \left[\frac{s-b+s-a}{(s-a)(s-b)} \right] - \Delta \left[\frac{1}{s-c} - \frac{1}{s} \right] = \Delta \left[\frac{c}{(s-a)(s-b)} - \frac{c}{s(s-c)} \right] = \Delta c \left[\frac{s(s-c) - (s-a)(s-b)}{s(s-a)(s-b)(s-c)} \right]$$

$$= \frac{\Delta c [s^2 - sc - s^2 + s(a+b) - ab]}{\Delta^2} = \frac{c [s(a+b-c) - ab]}{\Delta} = \frac{c[(a+b+c)(a+b-c) - 2ab]}{2\Delta}$$

$$= c \left[\frac{(a+b)^2 - c^2 - 2ab}{2\Delta} \right] = \frac{c(a^2 + b^2 - c^2)}{2\Delta}$$

$$\therefore \cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad \therefore \quad \text{L.H.S.} = \frac{c(2ab \cos C)}{2\Delta}$$

$$= \frac{ab \cos C}{\Delta} = \frac{4R \Delta \cos C}{\Delta} = 4R \cos C$$

$$(iii) \quad \therefore \quad \text{L.H.S.} = \frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{1}{\Delta^2} [s^2 + (s-a)^2 + (s-b)^2 + (s-c)^2]$$

$$= \frac{1}{\Delta^2} [4s^2 - 2s(a+b+c) + \Sigma a^2] = \frac{\Sigma a^2}{\Delta^2} = \text{R.H.S.}$$

$$(iv) \quad \therefore \quad \text{L.H.S.} = \left(\frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right)^2 = \frac{1}{\Delta^2} (s + s - a + s - b + s - c)^2 = 4 \frac{s^2}{\Delta^2} = \frac{4}{r^2}$$

$$\therefore \quad \text{R.H.S.} = \frac{4}{r} \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right) = \frac{4}{r} \cdot \frac{1}{\Delta} (s - a + s - b + s - c) = \frac{4}{r} \cdot \frac{s}{\Delta} = \frac{4}{r^2}$$

$$(v) \quad \therefore \quad \frac{bc - r_2 r_3}{r_1} = \frac{bc \left(1 - \frac{r_2 r_3}{bc} \right)}{r_1} = \frac{bc \left(1 - \frac{\Delta^2}{bc(s-b)(s-c)} \right)}{r_1}$$

$$= \frac{bc \left(1 - \frac{s(s-a)}{bc} \right)}{r_1} = \frac{bc \left(1 - \cos^2 \frac{A}{2} \right)}{r_1}$$

$$= \frac{bc \sin^2 \frac{A}{2}}{r_1} = \frac{bc (2 \sin \frac{A}{2} \cos \frac{A}{2})}{2s \sin \frac{A}{2}} = \frac{bc \sin A}{2s} = \frac{2\Delta}{2s} = r$$

similarly we can show that इसी प्रकार दर्शाया जा सकता है कि $\frac{ca - r_3 r_1}{r_2} = \frac{ab - r_1 r_2}{r_3} = r$

C-3. Show that the radii of the three escribed circles of a triangle are roots of the equation $x^3 - x^2(4R + r) + x s^2 - r s^2 = 0$.

प्रदर्शित कीजिए कि त्रिभुज के तीनों बहिर्वृत्तों की त्रिज्याएँ, समीकरण $x^3 - x^2(4R + r) + x s^2 - r s^2 = 0$ के मूल हैं।

Sol. $\therefore \quad \Sigma r_1 = r_1 + r_2 + r_3$

$$\therefore \quad r_1 + r_2 + r_3 - r = 4R \quad \therefore \quad \Sigma r_1 = 4R + r$$

$$\therefore \quad \Sigma r_1 r_2 = s^2$$



$$\therefore \Pi r_1 = \frac{\Delta^3}{(s-a)(s-b)(s-c)} = \frac{s\Delta^3}{\Delta^2} = s\Delta = rs^2$$

$\therefore$ equation having root r_1, r_2, r_3 is
 $x^3 - (4R + r)x^2 + (s^2)x - rs^2 = 0$.

$\therefore r_1, r_2, r_3$ मूलों वाली समीकरण होगी

C-4. The radii r_1, r_2, r_3 of escribed circles of a triangle ABC are in harmonic progression. If its area is 24 sq. cm and its perimeter is 24 cm, find the lengths of its sides.

यदि त्रिभुज ABC का क्षेत्रफल 24 वर्ग सेमी. तथा परिमाप 24 सेमी हो तथा साथ ही बहिर्वृत्तों की त्रिज्याएँ r_1, r_2, r_3 ह. श्रे. में हो, तो इसकी भुजाओं की लम्बाई ज्ञात कीजिए।

Ans. 6, 8, 10 cm सेमी

Sol. $\therefore \Delta = 24$ sq. cm वर्ग सेमी (i)

$$2s = 24 \Rightarrow s = 12 \text{ (ii)}$$

$\therefore r_1, r_2, r_3$ are in H.P. r_1, r_2, r_3 हरात्मक श्रेढी में है।

$\therefore \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}$ are in A.P. $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}$ समान्तर श्रेढी में है।

$\therefore \frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta}$ are in A.P. $\frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta}$ समान्तर श्रेढी में है।

$\therefore a, b, c$ are in A.P. a, b, c समान्तर श्रेढी में है $\Rightarrow 2b = a + c$

$$\therefore 2s = 24$$

$$\therefore a + b + c = 24$$

$$3b = 24$$

$$\therefore b = 8 \Rightarrow a + c = 16$$

But लेकिन $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$$\Rightarrow \Delta = \sqrt{12(12-a)(12-8)(12-c)} \Rightarrow 24 \times 24 = 12 \times (12-a) \times 4 \times (12-c)$$

$$\Rightarrow 2 \times 6 = 144 - 12(a+c) + ac \Rightarrow 12 = 144 - 192 + ac$$

$$\therefore ac = 60 \text{ and और } a + c = 16 \therefore a = 10, c = 6 \text{ or या } a = 6, c = 10 \text{ and और } b = 8$$

C-5. If the area of a triangle is 100 sq.cm, $r_1 = 10$ cm and $r_2 = 50$ cm, then find the value of $(b-a)$.

यदि एक त्रिभुज का क्षेत्रफल 100 वर्ग सेमी, $r_1 = 10$ सेमी एवं $r_2 = 50$ सेमी हो, तो $(b-a)$ का मान ज्ञात कीजिए।

Ans. 8

Sol. $\therefore D = 100 \text{ cm}^2$

$$r_1 = \frac{\Delta}{s-a} = 10 \Rightarrow s-a = 10 \text{(i)}$$

$$r_2 = \frac{\Delta}{s-b} = 50 \Rightarrow s-b = 2 \text{(ii)}$$

$$(i) - (ii)$$

$$\therefore b-a = 8$$

Section (D) Miscellaneous

खण्ड (D) विविध

D-1. If α, β, γ are the respective altitudes of a triangle ABC, prove that

यदि त्रिभुज ABC के शीर्षलम्ब क्रमशः α, β, γ हो, तो सिद्ध कीजिए कि -

$$(i) \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\cot A + \cot B + \cot C}{\Delta} \quad (ii) \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{2ab}{(a+b+c)\Delta} \cos^2 \frac{C}{2}$$



Sol. (i) $\therefore \alpha = \frac{2\Delta}{a}, \beta = \frac{2\Delta}{b}, \gamma = \frac{2\Delta}{c} \Rightarrow \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{a^2 + b^2 + c^2}{4\Delta^2}$

R.H.S. दायां पक्ष = $\frac{\cot A + \cot B + \cot C}{\Delta} = \frac{1}{\Delta} \left(\frac{bc \cos A}{2\Delta} + \frac{ca \cos B}{2\Delta} + \frac{ab \cos C}{2\Delta} \right)$

$= \frac{a^2 + b^2 + c^2}{4\Delta^2}$

$\therefore$ L.H.S. बायां पक्ष = R.H.S. दायां पक्ष

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\gamma} = \frac{a+b-c}{2\Delta} = \frac{2(s-c)}{2\Delta} = \frac{s-c}{\Delta}$

R.H.S. दायां पक्ष = $\frac{2ab}{(a+b+c)\Delta} \cdot \cos^2 \frac{C}{2} = \frac{2ab}{(2s)\Delta} \cdot \frac{s(s-c)}{ab}$

$= \frac{s-c}{\Delta}$

L.H.S. बायां पक्ष = R.H.S. दायां पक्ष

D-2. If in an acute angled $\triangle ABC$, line joining the circumcentre and orthocentre is parallel to side AC, then find the value of $\tan A \cdot \tan C$.

यदि किसी न्यूनकोण $\triangle ABC$ में, परिकेन्द्र तथा लम्बकेन्द्र को मिलाने वाली रेखा AC के समान्तर हो तो $\tan A \cdot \tan C$ का मान ज्ञात कीजिए।

Ans. 3

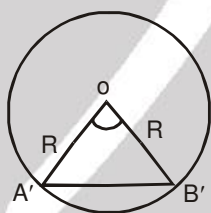
Sol. $\therefore$ line joining the circumcentre and orthocentre is parallel to side AC
 $\therefore$ परिकेन्द्र तथा लम्ब केन्द्र को मिलाने वाली रेखा, भुजा AC के समान्तर है,
 $\Rightarrow R \cos B = 2R \cos A \cos C \Rightarrow -\cos(A+C) = 2 \cos A \cos C$
 $\Rightarrow \sin A \sin C = 3 \cos A \cos C \Rightarrow \tan A \tan C = 3$

D-3. A regular hexagon & a regular dodecagon are inscribed in the same circle. If the side of the dodecagon is $(\sqrt{3} - 1)$, if the side of the hexagon is $\sqrt[4]{k}$, then find value of k.

एक समषट्भुज और एक समबारह भुज एक ही वृत्त के अन्दर निर्मित है, यदि बारहभुज की एक भुजा की लम्बाई $(\sqrt{3} - 1)$ हो, तो षट्भुज की भुजा की लम्बाई $\sqrt[4]{k}$ है, तो k का मान है—

Ans. $\sqrt{2}$

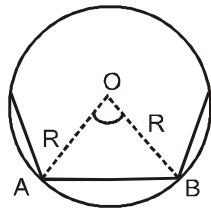
Sol.



For dodecagon $\angle A'OB' = \frac{2\pi}{12} = 30^\circ$

$\Rightarrow \angle OA'B' = \angle OB'A' = 75^\circ \Rightarrow \frac{R}{\sin 75^\circ} = \frac{\sqrt{3} - 1}{\sin 30^\circ}$

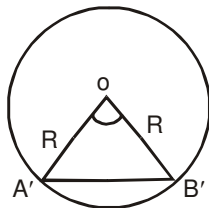
$\Rightarrow R = \frac{(\sqrt{3} - 1)(\sqrt{3} + 1)}{2\sqrt{2} \times \frac{1}{2}} \Rightarrow R = \sqrt{2}$



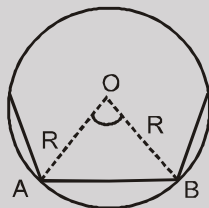
For hexagon $\angle AOB = \frac{2\pi}{6} = 60^\circ$

$\Rightarrow \Delta AOB$ is equilateral $\Rightarrow AB = R = \sqrt{2}$

Hindi समबाहु भुज हेतु $\angle A'OB' = \frac{2\pi}{12} = 30^\circ \Rightarrow \angle OA'B' = \angle OB'A' = 75^\circ$



$$\Rightarrow \frac{R}{\sin 75^\circ} = \frac{\sqrt{3}-1}{\sin 30^\circ} \Rightarrow R = \frac{(\sqrt{3}-1)(\sqrt{3}+1)}{2\sqrt{2} \times \frac{1}{2}} \Rightarrow R = \sqrt{2}$$



षट्भुज के लिए $\angle AOB = \frac{2\pi}{6} = 60^\circ$

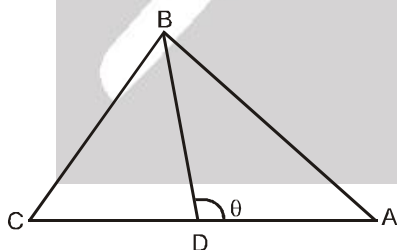
$\Rightarrow \Delta AOB$ समबाहु त्रिभुज है $\Rightarrow AB = R = \sqrt{2}$

D-4. If D is the mid point of CA in triangle ABC and Δ is the area of triangle, then show that

$$\tan(\angle ADB) = \frac{4\Delta}{a^2 - c^2}$$

यदि त्रिभुज ABC में CA का मध्य बिन्दु D तथा त्रिभुज का क्षेत्रफल Δ हो, तो प्रदर्शित कीजिए कि

$$\tan(\angle ADB) = \frac{4\Delta}{a^2 - c^2}$$



Sol.

Let माना $\angle ADB = \theta$

$\therefore$ we have to prove that हमें सिद्ध करना है कि $\tan \theta = \frac{4\Delta}{a^2 - c^2}$

if we apply m - n rule, then m - n प्रमेय का प्रयोग करने पर

$$(1 + 1) \cot \theta = 1 \cdot \cot C - 1 \cdot \cot A.$$



$$= \frac{\cos C}{\sin C} - \frac{\cos A}{\sin A} = \frac{ab \cos C}{2\Delta} - \frac{bc \cos A}{2\Delta} = \frac{1}{2\Delta} \left[\frac{b^2 + a^2 - c^2}{2} - \frac{b^2 + c^2 - a^2}{2} \right]$$

$$= \frac{1}{4\Delta} [2(a^2 - c^2)]$$

$$\therefore 2 \cot \theta = \frac{a^2 - c^2}{2\Delta} \quad \therefore \tan \theta = \frac{4\Delta}{a^2 - c^2}$$

Exercise-2

Marked questions are recommended for Revision.

चिह्नित प्रश्न दोहराने योग्य प्रश्न है।

PART-I (OBJECTIVE QUESTIONS)

Section (A) : Sine rule, Cosine rule, Napier's Analogy, Projection rule

खण्ड (A) : ज्या नियम, कोज्या नियम, स्पर्शज्या नियम या नेपियर एनालोजि, प्रक्षेप नियम

- A-1.** In a $\triangle ABC$, $A : B : C = 3 : 5 : 4$. Then $a + b + c \sqrt{2}$ is equal to
 त्रिभुज ABC में, $A : B : C = 3 : 5 : 4$ हो, तो $a + b + c \sqrt{2}$ का मान है –
 (A) $2b$ (B) $2c$ (C*) $3b$ (D) $3a$

Sol. $\therefore A : B : C = 3 : 5 : 4 \quad \therefore A = 45^\circ, B = 75^\circ, C = 60^\circ$
 $\therefore$ from Sine - rule ज्या नियम के अनुसार
 $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \Rightarrow \frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{c}{\frac{\sqrt{3}}{2}} = k \quad (\because \sin 75^\circ = \sin(45^\circ + 30^\circ))$
 $\therefore a = \frac{k}{\sqrt{2}}, b = \left(\frac{\sqrt{3}+1}{2\sqrt{2}} \right) k$ and $c = \frac{k\sqrt{3}}{2}$
 $\therefore a + b + c\sqrt{2} = \frac{k}{\sqrt{2}} + \left(\frac{\sqrt{3}+1}{2\sqrt{2}} \right) k + \left(\frac{k\sqrt{3}}{2} \right) \sqrt{2} = \frac{k}{2\sqrt{2}} [2 + (\sqrt{3}+1) + 2\sqrt{3}] = \frac{3k(\sqrt{3}+1)}{2\sqrt{2}} = 3b$

- A-2*.** In a triangle ABC, the altitude from A is not less than BC and the altitude from B is not less than AC. The triangle is
 (A*) right angled (B*) isosceles (C) obtuse angled (D) equilateral
 ABC में A से शीर्षलम्ब, BC से कम नहीं है तथा B से शीर्षलम्ब AC से कम नहीं है, तब त्रिभुज है –
 (A) समकोण (B) समद्विबाहु (C) अधिक कोण (D) समबाहु

Sol. Given $c \sin B \geq a$
 $\Rightarrow \sin C \sin B \geq \sin A$
 and $a \sin C \geq b \Rightarrow \sin C \sin A \geq \sin B$
 $\Rightarrow \sin C \cdot \sin A \geq \sin B \geq \frac{\sin A}{\sin C}$

$$\therefore \sin^2 C \geq 1 \Rightarrow \sin C = 1 \Rightarrow \angle C \text{ is } \frac{\pi}{2}$$

- A-3.** If in a $\triangle ABC$, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, then the triangle is :
 (A) right angled (B) isosceles (C*) equilateral (D) obtuse angled
 यदि त्रिभुज ABC में, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ हो, तो त्रिभुज है –





- (A) समकोण त्रिभुज (B) समद्विबाहु त्रिभुज (C) समबाहु त्रिभुज (D) अधिक कोण त्रिभुज

Sol. given दिया है $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ (i)

$\therefore a = k \sin A, b = k \sin B, c = k \sin C \quad \therefore$ (i) becomes (i) में प्रयोग करने पर

$\frac{\cot A}{k} = \frac{\cot B}{k} = \frac{\cot C}{k} \quad \therefore A = B = C$

$\therefore \triangle ABC$ is an equilateral triangle (त्रिभुज ABC एक समबाहु त्रिभुज होगा।)

A-4. In a $\triangle ABC$ $\frac{bc \sin^2 A}{\cos A + \cos B \cos C}$ is equal to

- (A) $b^2 + c^2$ (B) bc (C*) a^2 (D) $a^2 + bc$

त्रिभुज ABC में, $\frac{bc \sin^2 A}{\cos A + \cos B \cos C}$ बराबर है—

- (A) $b^2 + c^2$ के (B) bc के (C*) a^2 के (D) $a^2 + bc$ के

Sol. $\therefore \frac{bc \sin^2 A}{\cos A + \cos B \cos C} = \frac{k^2 \sin B \sin C \sin^2 A}{-\cos(B+C) + \cos B \cos C} = \frac{k^2 \sin B \sin C \sin^2 A}{\sin B \sin C} = k^2 \sin^2 A = a^2.$

A-5. Given a triangle $\triangle ABC$ such that $\sin^2 A + \sin^2 C = 1001 \sin^2 B$. Then the value of $\frac{2(\tan A + \tan C) \cdot \tan^2 B}{\tan A + \tan B + \tan C}$ is

दिए गए त्रिभुज ABC में $\sin^2 A + \sin^2 C = 1001 \sin^2 B$. तब $\frac{2(\tan A + \tan C) \cdot \tan^2 B}{\tan A + \tan B + \tan C}$ बराबर है —

- (A) $\frac{1}{2000}$ (B) $\frac{1}{1000}$ (C) $\frac{1}{500}$ (D*) $\frac{1}{250}$

Sol. $\sin^2 A + \sin^2 C = 1001 \sin^2 B$

$\Rightarrow a^2 + c^2 = 1001 b^2$ (using sine rule)

Now, $\frac{2(\tan A + \tan C) \cdot \tan^2 B}{\tan A + \tan B + \tan C} = \frac{2(\tan A + \tan C) \cdot \tan^2 B}{\tan A \cdot \tan B \tan C} = 2 \left(\frac{\cot A + \cot C}{\cot B} \right)$

$= \sin B \frac{2(\cos A \sin C + \sin A \cos C)}{\sin A \cdot \sin C \cdot \cos B} = \frac{2 \sin(\pi - B) \cdot \sin B}{\sin A \sin C \cos B} = \frac{2 \sin^2 B}{\sin A \sin C \cos B}$

$= \frac{2 \times 2b^2}{2ac \cdot \cos B} = \frac{2 \times 2b^2}{a^2 + c^2 - b^2} = \frac{2 \times 2b^2}{1000b^2} = \frac{1}{250}$

A-6. If in a triangle ABC, $(a + b + c)(b + c - a) = k \cdot bc$, then :

- (A) $k < 0$ (B) $k > 6$ (C*) $0 < k < 4$ (D) $k > 4$

यदि त्रिभुज ABC में $(a + b + c)(b + c - a) = k \cdot bc$ हो, तो —

- (A) $k < 0$ (B) $k > 6$ (C*) $0 < k < 4$ (D) $k > 4$

Sol. $\therefore (a + b + c)(b + c - a) = kbc \quad \Rightarrow (b + c)^2 - a^2 = kbc$

$b^2 + c^2 - a^2 = (k - 2)bc \quad \Rightarrow \frac{b^2 + c^2 - a^2}{2bc} = \frac{k - 2}{2} = \cos A$

$\therefore$ In a $\triangle ABC$ में $-1 < \cos A < 1 \quad \therefore -1 < \frac{k - 2}{2} < 1$

$0 < k < 4$

A-7. In a triangle ABC, $a : b : c = 4 : 5 : 6$. Then $3A + B$ equals to :

त्रिभुज ABC में, $a : b : c = 4 : 5 : 6$ हो, तो $3A + B$ का मान है—

- (A) $4C$ (B) 2π (C) $\pi - C$ (D*) π

Sol. $\therefore a : b : c = 4 : 5 : 6 \quad \therefore a = 4k, b = 5k, c = 6k$

$\therefore \cos B = \frac{c^2 + a^2 - b^2}{2ac} = \frac{9}{16} \quad \therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{25 + 36 - 16}{2 \times 5 \times 6} = \frac{3}{4}$





$$\begin{aligned}\therefore \cos 3A &= 4 \cos^3 A - 3 \cos A \\ &= 4 \times \frac{27}{64} - 3 \times \frac{3}{4} = \frac{27}{16} - \frac{9}{4} = \frac{27-36}{16} = \frac{-9}{16} \\ \cos 3A &= -\cos B = \cos (\pi - B) \\ \therefore 3A + B &= \pi\end{aligned}$$

A-8. The distance between the middle point of BC and the foot of the perpendicular from A is ($b > c$) :

- (A) $\frac{-a^2 + b^2 + c^2}{2a}$ (B*) $\frac{b^2 - c^2}{2a}$ (C) $\frac{b^2 + c^2}{\sqrt{bc}}$ (D) $\frac{b^2 + c^2}{2a}$
- भुजा BC के मध्य बिन्दु एवं शीर्ष A से डाले गए लम्ब के पाद के मध्य दूरी है ($b > c$)
- (A) $\frac{-a^2 + b^2 + c^2}{2a}$ (B*) $\frac{b^2 - c^2}{2a}$ (C) $\frac{b^2 + c^2}{\sqrt{bc}}$ (D) $\frac{b^2 + c^2}{2a}$

Sol. $\therefore ED = \frac{a}{2} - c \cos B$

$$= \frac{a}{2} - c \left(\frac{a^2 + c^2 - b^2}{2ac} \right)$$

$$= \frac{a}{2} - \left(\frac{a^2 + c^2 - b^2}{2a} \right) = \frac{a^2 - a^2 - c^2 + b^2}{2a} = \frac{b^2 - c^2}{2a}$$

A-9*. If in a triangle ABC, $\cos A \cos B + \sin A \sin B \sin C = 1$, then the triangle is
 (A*) isosceles (B*) right angled (C) equilateral (D) None of these
 यदि त्रिभुज ABC में $\cos A \cos B + \sin A \sin B \sin C = 1$ हो, तो त्रिभुज है -
 (A) समद्विबाहु त्रिभुज (B) समकोण त्रिभुज (C) समबाहु त्रिभुज (D) इनमें से कोई नहीं

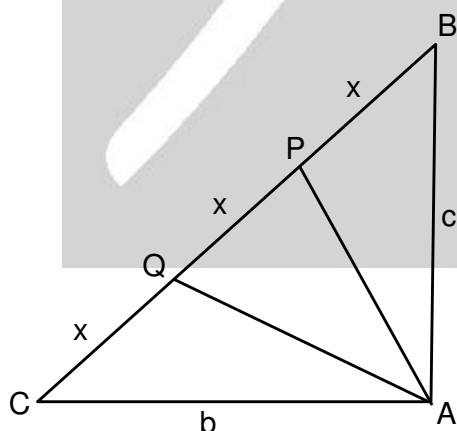
Sol. $\therefore \sin C = \frac{1 - \cos A \cos B}{\sin A \sin B} \leq 1 \Rightarrow \cos (A - B) \geq 1 \Rightarrow \cos (A - B) = 1$

$$\Rightarrow A - B = 0 \Rightarrow A = B \therefore \sin C = \frac{1 - \cos^2 A}{\sin^2 A} = 1 \Rightarrow C = 90^\circ$$

A-10. Triangle ABC is right angle at A. The points P and Q are on hypotenuse BC such that $BP = PQ = QC$. If $AP = 3$ and $AQ = 4$, then length BC is equal to
 त्रिभुज ABC में A पर समकोण है। बिन्दु P और Q कर्ण BC पर इस प्रकार है कि $BP = PQ = QC$. यदि $AP = 3$ और $AQ = 4$ तब BC की लम्बाई है -

- (A*) $3\sqrt{5}$ (B) $5\sqrt{3}$ (C) $4\sqrt{5}$ (D) 7

Sol.



Let $BP = PQ = QC = x$
 In $\triangle ABP$, using cosine rule
 $9 = c^2 + x^2 - 2cx \cos B$





$$\text{But } \cos B = \frac{c}{3x} \quad \dots\dots(1)$$

$$\Rightarrow 9 = x^2 + \frac{c^2}{3}$$

similarly using cosine rule in $\triangle ACQ$, we get

$$16 = x^2 + \frac{b^2}{3} \quad \dots\dots(2)$$

$$\text{Adding (1) and (2), we get } 25 = 2x^2 + \left(\frac{b^2 + c^2}{3} \right)$$

$$\therefore 25 = 2x^2 + \frac{(3x)^2}{3}$$

$$\therefore 25 = 2x^2 + 3x^2$$

$$\therefore x^2 = 5$$

$$\therefore BC = 3x = 3\sqrt{5}$$

A-11. In $\triangle ABC$, $bc = 2b^2 \cos A + 2c^2 \cos A - 4bc \cos^2 A$, then $\triangle ABC$ is

(A*) isosceles but not necessarily equilateral

(B) equilateral

(C) right angled but not necessarily isosceles

(D) right angled isosceles

$\triangle ABC$ में $bc = 2b^2 \cos A + 2c^2 \cos A - 4bc \cos^2 A$, तब $\triangle ABC$ है -

(A) समद्विबाहु परन्तु आवश्यक नहीं कि समबाहु

(B) समबाहु

(C) समकोण त्रिभुज परन्तु आवश्यक नहीं कि समद्विबाहु

(D) समद्विबाहु समकोण त्रिभुज

Sol. $bc = 2b^2 \cos A + 2c^2 \cos A - 4bc \cos^2 A$

$$\Rightarrow bc = 2 \cos A (b^2 + c^2 - 2bc \cos A)$$

$$\Rightarrow bc = (2 \cos A) a^2$$

$$\Rightarrow \frac{bc}{2a^2} = \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow b^2 c^2 = a^2 (b^2 + c^2 - a^2)$$

$$\Rightarrow (a^2 - b^2) (a^2 - c^2) = 0$$

Thus, triangle is isosceles

Section (B) Trigonometric ratios of Half Angles, Area of triangle and circumradius

खण्ड (B) अर्द्धकोण, त्रिभुज का क्षेत्रफल और परित्रिज्या में त्रिकोणमितिय सम्बन्ध

B-1. If in a triangle ABC, right angle at B, $s - a = 3$ and $s - c = 2$, then

त्रिभुज ABC जो B पर समकोण है, में $s - a = 3$ एवं $s - c = 2$ हो, तो -

(A) $a = 2, c = 3$

(B*) $a = 3, c = 4$

(C) $a = 4, c = 3$

(D) $a = 6, c = 8$

Sol. $\therefore s - a = 3$

...(1) and $s - c = 2$

...(2)

by (1) - (2), we get

$$c - a = 1$$

$$(1) + (2), \text{ we get } 2s - a - c = 5 \quad \Rightarrow \quad b = 5$$

$\therefore \triangle ABC$ is right angled at B

$$\therefore a^2 + c^2 = 25 \quad \dots(3)$$

$$\therefore (c - a)^2 + 2ac = 25$$

$$ac = 12 \quad \dots(4)$$

$$\therefore a(1 + a) = 12 \Rightarrow a^2 + a - 12 = 0$$

$$\Rightarrow (a + 4) (a - 3) = 0$$

$$\Rightarrow a = 3 \text{ and } c = 4.$$





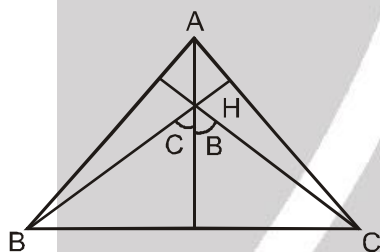
Hindi $\therefore s - a = 3 \dots(1)$ तथा $s - c = 2 \dots(2)$
 समीकरण (1) में से (2) को घटाने पर
 $c - a = 1$
 समीकरण (1) में (2) को जोड़ने पर
 $2s - a - c = 5 \Rightarrow b = 5$
 $\therefore \Delta ABC$ शीर्ष B पर समकोण है, $\therefore a^2 + c^2 = 25 \dots(3)$
 $\therefore (c - a)^2 + 2ac = 25$
 $ac = 12 \dots(4)$
 $\therefore a(1 + a) = 12 \Rightarrow a^2 + a - 12 = 0 \Rightarrow (a + 4)(a - 3) = 0 \Rightarrow a = 3$ और $c = 4$.

B-2. If in a triangle ABC, $b \cos^2 \frac{A}{2} + a \cos^2 \frac{B}{2} = \frac{3}{2}c$, then a, c, b are :
 (A*) in A.P. (B) in G.P. (C) in H.P. (D) None
 यदि किसी त्रिभुज ABC में, $b \cos^2 \frac{A}{2} + a \cos^2 \frac{B}{2} = \frac{3}{2}c$, हो, तो a, c, b हैं -
 (A) समान्तर श्रेढ़ी में (B) गुणोत्तर श्रेढ़ी में (C) हरात्मक श्रेढ़ी में (D) इनमें से कोई नहीं।

Sol. $\therefore b \cos^2 \frac{A}{2} + a \cos^2 \frac{B}{2} = \frac{3}{2}c \Rightarrow b \frac{s(s-a)}{bc} + a \frac{s(s-b)}{ac} = \frac{3}{2}c$
 $\Rightarrow \frac{s}{c} [s - a + s - b] = \frac{3}{2}c \Rightarrow \frac{s}{c} \times c = \frac{3}{2}c \Rightarrow \frac{a+b+c}{2} = \frac{3c}{2} \Rightarrow a + b = 2c$
 $\Rightarrow a, c, b$ are in A.P. a, c, b समान्तर श्रेढ़ी में है।

B-3. If H is the orthocentre of a triangle ABC, then the radii of the circle circumscribing the triangles BHC, CHA and AHB are respectively equal to :
 यदि त्रिभुज ABC का लम्बकेन्द्र H हो, तो त्रिभुजों BHC, CHA एवं AHB के परिगत वृत्तों की त्रिज्याएँ क्रमशः हैं -
 (A*) R, R, R (B) $\sqrt{2}R, \sqrt{2}R, \sqrt{2}R$ (C) 2R, 2R, 2R (D) $\frac{R}{2}, \frac{R}{2}, \frac{R}{2}$

Sol.



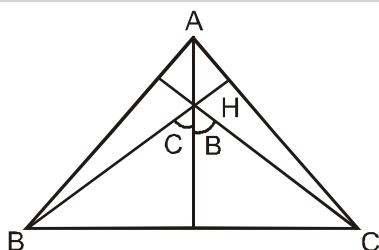
In ΔHBC is we apply Sine-rule, then we get

$$\frac{BC}{\sin(B+C)} = 2R'$$

$$\frac{a}{\sin A} = 2R' \Rightarrow 2R = 2R' \Rightarrow R = R'$$

$\therefore$ circumradius of ΔHBC (i.e. R') = R
 Similarly we can prove for ΔHCA and ΔHAB .

Hindi





त्रिभुज HBC में ज्या नियम के प्रयोग से

$$\frac{BC}{\sin(B+C)} = 2R'$$

$$\frac{a}{\sin A} = 2R' \Rightarrow 2R = 2R' \Rightarrow R = R'$$

$\therefore \Delta HBC$ की परित्रिज्या (अर्थात् R') = R

इसी प्रकार ΔHCA और ΔHAB के लिये भी सिद्ध किया जा सकता है।

B-4. In a ΔABC if $b + c = 3a$, then $\cot \frac{B}{2} \cdot \cot \frac{C}{2}$ has the value equal to:

त्रिभुज ABC में यदि $b + c = 3a$ हो, तो $\cot \frac{B}{2} \cdot \cot \frac{C}{2}$ का मान है -

- (A) 4 (B) 3 (C*) 2 (D) 1

Sol. $\therefore \cot \frac{B}{2} \cot \frac{C}{2} = \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \frac{s}{s-a} = \frac{2s}{2s-2a}$
 $= \frac{a+b+c}{b+c-a} = \frac{4a}{2a} = 2 \quad (\because b+c=3a)$

B-5. In a ΔABC , $A = \frac{2\pi}{3}$, $b - c = 3\sqrt{3}$ cm and area (ΔABC) = $\frac{9\sqrt{3}}{2}$ cm². Then 'a' is

- (A) $6\sqrt{3}$ cm (B*) 9 cm (C) 18 cm (D) 7 cm

त्रिभुज ABC में $A = \frac{2\pi}{3}$, $b - c = 3\sqrt{3}$ सेमी एवं क्षेत्रफल (ΔABC) = $\frac{9\sqrt{3}}{2}$ वर्ग सेमी हो, तो 'a' का मान है-

- (A) $6\sqrt{3}$ सेमी (B) 9 सेमी (C) 18 सेमी (D) 7 सेमी

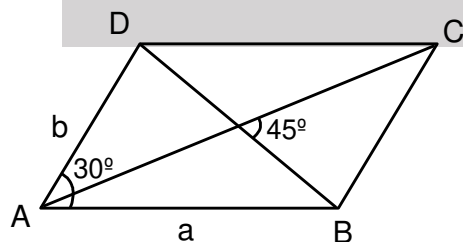
Sol. $\therefore A = \frac{2\pi}{3}$, $b - c = 3\sqrt{3}$ and तथा Area (क्षेत्रफल) = $\frac{9\sqrt{3}}{2}$ cm²
 $\therefore \Delta = \frac{1}{2}bc \sin A \Rightarrow \frac{9\sqrt{3}}{2} = \frac{1}{2}bc \sin \frac{2\pi}{3} \Rightarrow bc = 18$
 $\therefore \cos \frac{2\pi}{3} = \frac{b^2 + c^2 - a^2}{2bc} = -\frac{1}{2} \Rightarrow \frac{(b-c)^2 + 2bc - a^2}{2bc} = -\frac{1}{2} \Rightarrow a = 9$

B-6.* The diagonals of a parallelogram are inclined to each other at an angle of 45° , while its sides a and b ($a > b$) are inclined to each other at an angle of 30° , then the value of $\frac{a}{b}$ is

समान्तर चतुर्भुज के विकर्ण एक दूसरे को 45° पर झुके हुए हैं जबकि इसकी भुजाएँ a और b ($a > b$) एक दूसरे से 30° , कोण पर झुकी है तब $\frac{a}{b}$ का मान है -

- (A*) $2\cos 36^\circ$ (B) $\sqrt{\frac{3+\sqrt{5}}{4}}$ (C) $\frac{3+\sqrt{5}}{4}$ (D*) $\frac{\sqrt{5}+1}{2}$

Sol.



Let $AC = d_1$ and $BD = d_2$



$$\text{Area of parallelogram is } \frac{1}{2} d_1 d_2 \sin 45^\circ = 2 \left(\frac{1}{2} ab \sin 30^\circ \right) \Rightarrow d_1 d_2 = \sqrt{2} ab \quad \dots(i)$$

$$\text{where } d_1^2 = a^2 + b^2 - 2ab \cos 150^\circ = a^2 + b^2 + \sqrt{3} ab$$

$$\text{and } d_2^2 = a^2 + b^2 - 2ab \cos 30^\circ = a^2 + b^2 - \sqrt{3} ab$$

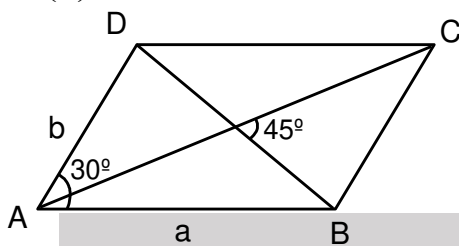
$$\Rightarrow d_1^2 d_2^2 = (a^2 + b^2)^2 - 3a^2 b^2$$

$$\Rightarrow 2a^2 b^2 = (a^2 + b^2)^2 - 3a^2 b^2$$

$$\Rightarrow a^4 + b^4 - 3a^2 b^2 = 0$$

$$\Rightarrow \left(\frac{a}{b} \right)^4 - 3 \left(\frac{a}{b} \right)^2 + 1 = 0 \Rightarrow \left(\frac{a}{b} \right)^2 = \frac{3 + \sqrt{5}}{2} \quad (\text{as } a > b)$$

$$\Rightarrow \left(\frac{a}{b} \right)^2 = \frac{(\sqrt{5} + 1)^2}{4} \Rightarrow \frac{a}{b} = \frac{\sqrt{5} + 1}{2}$$



Hindi.

माना $AC = d_1$ तथा $BD = d_2$

$$\text{समान्तर चतुर्भुज क्षेत्रफल} = \frac{1}{2} d_1 d_2 \sin 45^\circ = 2 \left(\frac{1}{2} ab \sin 30^\circ \right) \Rightarrow d_1 d_2 = \sqrt{2} ab \quad \dots(i)$$

$$\text{जहाँ } d_1^2 = a^2 + b^2 - 2ab \cos 150^\circ = a^2 + b^2 + \sqrt{3} ab$$

$$\text{और } d_2^2 = a^2 + b^2 - 2ab \cos 30^\circ = a^2 + b^2 - \sqrt{3} ab$$

$$\Rightarrow d_1^2 d_2^2 = (a^2 + b^2)^2 - 3a^2 b^2$$

$$\Rightarrow 2a^2 b^2 = (a^2 + b^2)^2 - 3a^2 b^2 \Rightarrow a^4 + b^4 - 3a^2 b^2 = 0$$

$$\Rightarrow \left(\frac{a}{b} \right)^4 - 3 \left(\frac{a}{b} \right)^2 + 1 = 0 \Rightarrow \left(\frac{a}{b} \right)^2 = \frac{3 + \sqrt{5}}{2} \quad (\text{चूँकि } a > b)$$

$$\Rightarrow \left(\frac{a}{b} \right)^2 = \frac{(\sqrt{5} + 1)^2}{4} \Rightarrow \frac{a}{b} = \frac{\sqrt{5} + 1}{2}$$

B-7. If in a $\triangle ABC$, $\Delta = a^2 - (b - c)^2$, then $\tan A$ is equal to
यदि किसी त्रिभुज ABC में, $\Delta = a^2 - (b - c)^2$ हो, तो $\tan A$ का मान है -

(A) 15/16 (B*) 8/15 (C) 8/17 (D) 1/2

Sol. $\Delta = (a + b - c)(a - b + c)$

$$\Delta = 4(s - c)(s - b) \Rightarrow \frac{\Delta}{(s - b)(s - c)} = \frac{1}{4} \therefore \tan \frac{A}{2} = \frac{1}{4}$$

$$\therefore \tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}} \Rightarrow \tan A = \frac{8}{15}$$

B-8*. If in a $\triangle ABC$, $a = 5$, $b = 4$ and $\cos(A - B) = \frac{31}{32}$, then

(A*) $c = 6$ (B*) $\sin A = \left(\frac{5\sqrt{7}}{16} \right)$ (C*) area of $\triangle ABC = \frac{15\sqrt{7}}{4}$ (D) $c = 8$

यदि किसी $\triangle ABC$ के लिए, $a = 5$, $b = 4$ और $\cos(A - B) = \frac{31}{32}$ हो, तो



$$(A^*) c = 6 \quad (B^*) \sin A = \left(\frac{5\sqrt{7}}{16} \right) \quad (C^*) \Delta ABC \text{ का क्षेत्रफल} = \frac{15\sqrt{7}}{4} \quad (D) c = 8$$

Sol. (A) $\therefore \tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right) \cot \frac{C}{2} \dots\dots(i)$

$$\therefore \tan^2\left(\frac{A-B}{2}\right) = \frac{1-\cos(A-B)}{1+\cos(A-B)} = \frac{1-\frac{31}{32}}{1+\frac{31}{32}} = \frac{1}{63}$$

$$\therefore \tan\left(\frac{A-B}{2}\right) = \frac{1}{3\sqrt{7}} \quad \therefore a = 5 \text{ and } b = 4$$

$\therefore$ from equation (i), we get समीकरण (i) से

$$\frac{1}{3\sqrt{7}} = \left(\frac{5-4}{5+4}\right) \cot \frac{C}{2} \Rightarrow \frac{1}{3\sqrt{7}} = \frac{1}{9} \cot \frac{C}{2} \Rightarrow \cot \frac{C}{2} = \frac{3}{\sqrt{7}}$$

$$\therefore \cos C = \frac{1-\tan^2 C/2}{1+\tan^2 C/2} = \frac{1-7/9}{1+7/9} = \frac{2}{16} = \frac{1}{8}$$

$$\therefore \cos C = \frac{b^2 + a^2 - c^2}{2ab} \Rightarrow c^2 = a^2 + b^2 - 2ab \cos C \Rightarrow c = 6$$

(B), (C) $\therefore \text{Area क्षेत्रफल} = \frac{1}{2} ab \sin C \quad \therefore \cos C = \frac{1}{8} \Rightarrow \sin C = \sqrt{1-\frac{1}{64}} = \frac{3\sqrt{7}}{8}$

$$\text{Area} = \frac{1}{2} \times 5 \times 4 \times \frac{3\sqrt{7}}{8}$$

Area क्षेत्रफल = sq. unit. वर्ग इकाई $\therefore$ From Sine rule ज्या नियम के प्रयोग से

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \sin A = \frac{a \sin C}{c} = \frac{5 \times 3\sqrt{7}}{6 \times 8} \therefore \sin A = \frac{5\sqrt{7}}{16}$$

B-9. If R denotes circumradius, then in ΔABC , $\frac{b^2 - c^2}{2aR}$ is equal to

(A) $\cos(B-C)$ (B*) $\sin(B-C)$ (C) $\cos B - \cos C$ (D) $\sin(B+C)$

यदि R परित्रिज्या को प्रदर्शित करता हो, तो त्रिभुज ABC में $\frac{b^2 - c^2}{2aR}$ का मान है -

(A) $\cos(B-C)$ (B) $\sin(B-C)$ (C) $\cos B - \cos C$ (D) $\sin(B+C)$

Sol. $\therefore \frac{b^2 - c^2}{2aR} = \frac{4R^2(\sin^2 B - \sin^2 C)}{2 \cdot 2R \sin A \cdot R} = \frac{\sin(B+C) \cdot \sin(B-C)}{\sin A} = \sin(B-C)$

B-10*. Which of the following holds good for any triangle ABC?

किसी भी ΔABC के लिये, निम्न में से सत्य है :-

(A*) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$ (B*) $\frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c} = \frac{3}{2R}$

(C) $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ (D) $\frac{\sin 2A}{a^2} = \frac{\sin 2B}{b^2} = \frac{\sin 2C}{c^2}$

Sol. (A) $\therefore \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{b^2 + c^2 - a^2}{2abc} + \frac{c^2 + a^2 - b^2}{2abc} + \frac{a^2 + b^2 - c^2}{2abc}$

$$= \frac{a^2 + b^2 + c^2}{2abc}$$

(B) $\therefore \frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c} = \frac{a}{2R \cdot a} + \frac{b}{2R \cdot b} + \frac{c}{2R \cdot c} = \frac{3}{2R}$



$$(C) \quad \frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c} \Rightarrow \cot A = \cot B = \cot C \Rightarrow A = B = C$$

true for equilateral triangle only केवल समबाहु त्रिभुज के लिए सत्य है

$$(D) \quad \frac{\sin 2A}{a^2} = \frac{\sin 2B}{b^2} = \frac{\sin 2C}{c^2}$$

$$\Rightarrow \frac{2 \sin A \cos A}{k^2 \sin^2 A} = \frac{2 \sin B \cos B}{k^2 \sin^2 B} = \frac{2 \sin C \cos C}{k^2 \sin^2 C}$$

$$\Rightarrow \cot A = \cot B = \cot C \Rightarrow A = B = C \Rightarrow \text{true for equilateral triangle only}$$

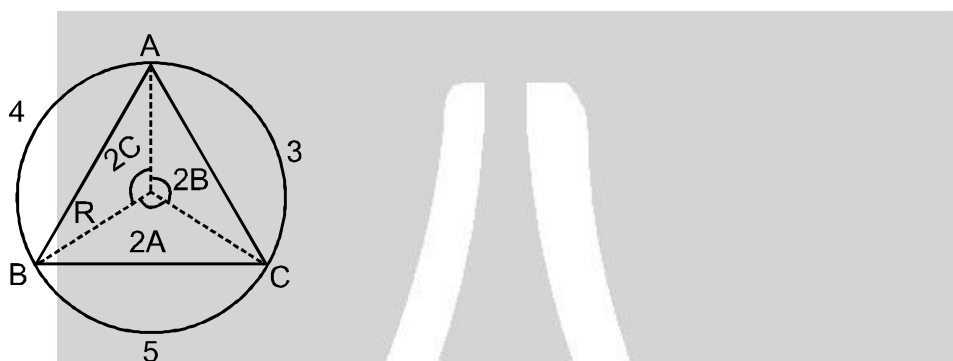
केवल समबाहु त्रिभुज के लिए सत्य है

B-11. A triangle is inscribed in a circle. The vertices of the triangle divide the circle into three arcs of length 3, 4 and 5 units. Then area of the triangle is equal to:

किसी वृत्त में एक त्रिभुज बनाया जाता है। त्रिभुज के शीर्ष वृत्त को 3 इकाई, 4 इकाई एवं 5 इकाई के तीन चापों में विभाजित करते हो, तो त्रिभुज का क्षेत्रफल है -

(A*) $\frac{9\sqrt{3}(1+\sqrt{3})}{\pi^2}$ (B) $\frac{9\sqrt{3}(\sqrt{3}-1)}{\pi^2}$ (C) $\frac{9\sqrt{3}(1+\sqrt{3})}{2\pi^2}$ (D) $\frac{9\sqrt{3}(\sqrt{3}-1)}{2\pi^2}$

Sol.



$$\text{angle} = \frac{\text{arc}}{\text{radius}} \dots\dots(1)$$

$$\text{कोण} = \frac{\text{चाप}}{\text{त्रिज्या}}$$

$$\therefore 4 + 5 + 3 = 2\pi R \Rightarrow R = 6/\pi$$

$$\therefore 2A = \frac{5}{R} = \frac{5\pi}{6},$$

$$2B = \frac{3}{R} = \frac{\pi}{2} \text{ and और}$$

$$2C = \frac{4}{R} = \frac{2\pi}{3}$$

$$\text{Area of } \triangle ABC \text{ का क्षेत्रफल} = \frac{1}{2} R^2 \left[\sin \frac{2\pi}{3} + \sin \frac{5\pi}{6} + \sin \frac{\pi}{2} \right]$$

$$= \frac{R^2}{2} \left[\frac{\sqrt{3}}{2} + \frac{1}{2} + 1 \right] = \frac{R^2}{2} \left[\frac{\sqrt{3}+3}{2} \right] = \frac{\sqrt{3}(\sqrt{3}+1)}{4} \times \frac{36}{\pi^2} = \frac{9\sqrt{3}(\sqrt{3}+1)}{\pi^2}$$

B-12. In a $\triangle ABC$, $a = 1$ and the perimeter is six times the arithmetic mean of the sines of the angles. Then measure of $\angle A$ is

त्रिभुज ABC में, $a = 1$ और परिमाप, कोणों की ज्याओं के समान्तर माध्य का छः गुना हो, तो कोण A का मान है -

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C*) $\frac{\pi}{6}$ (D) $\frac{\pi}{4}$

Sol. $a = 1$



$$\therefore 2s = 6 \left(\frac{\sin A + \sin B + \sin C}{3} \right)$$

$$2s = 2 \left(\frac{a+b+c}{2R} \right)$$

$$R = 1$$

$$\therefore \frac{a}{\sin A} = 2R \Rightarrow \sin A = \frac{1}{2}$$

$$A = \frac{\pi}{6}$$

B-13*. Three equal circles of radius unity touches one another. Radius of the circle touching all the three circles is :

इकाई त्रिज्या के तीन समान वृत्त एक दूसरे को स्पर्श करते हैं। तीनों वृत्तों को स्पर्श करने वाले वृत्त की त्रिज्या है –

(A*) $\frac{2-\sqrt{3}}{\sqrt{3}}$

(B) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$

(C*) $\frac{2+\sqrt{3}}{\sqrt{3}}$

(D) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}}$

Sol. Let the radius of the inner circle be x

माना दिये गये तीनों वृत्तों को स्पर्श करने वाले छोटे वृत्त (चित्रानुसार) की त्रिज्या x है।

$$\therefore \cos 30^\circ = \frac{1}{x+1} = \frac{\sqrt{3}}{2}$$

$$x+1 = \frac{2}{\sqrt{3}}$$

$$x = \frac{2-\sqrt{3}}{\sqrt{3}}$$

$\therefore$ radius of other (shaded) circle

दिये गये तीनों वृत्तों को स्पर्श करने वाले बड़े वृत्त (चित्रानुसार) की त्रिज्या

$$= 2 + x = 2 + \frac{2-\sqrt{3}}{\sqrt{3}} = \frac{2+\sqrt{3}}{\sqrt{3}}$$

B-14. Triangle ABC is isosceles with $AB = AC$ and $BC = 65$ cm. P is a point on BC such that the perpendicular distances from P to AB and AC are 24 cm and 36 cm, respectively. The area of triangle ABC (in sq. cm is)

त्रिभुज ABC समद्विबाहु त्रिभुज है जिसमें $AB = AC$ तथा $BC = 65$ cm है। बिन्दु P, BC पर इस प्रकार है कि P से AB व AC पर क्रमशः लम्बवत् दूरियाँ 24 सेमी और 36 सेमी है। त्रिभुज ABC का क्षेत्रफल (वर्ग सेमी. में)

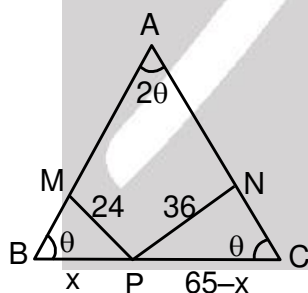
(A) 1254

(B) 1950

(C*) 2535

(D) 5070

Sol.



$$A = \frac{1}{2} b^2 \sin 2\theta = b^2 \sin \theta \cos \theta$$

$$\text{Now, } \frac{x}{24} = \frac{65-x}{36}$$

$$(\because \triangle PMB \cong \triangle PNC)$$

$$\text{or } 60x = (24)(65) \text{ or } x = 26$$



$$\therefore \sin \theta = \frac{12}{13} \text{ and } \cos \theta = \frac{5}{13}$$

$$\text{Again, } \frac{b}{\sin \theta} = \frac{65}{\sin 2\theta}$$

$$\text{or } b = \frac{65}{2 \cos \theta} = \frac{(65)(13)}{(2)(5)} = \frac{13^2}{2}$$

From eq. (i) we get

$$A = \frac{13^4}{4} \times \frac{12}{13} \times \frac{5}{13} = (169)(15) = 2535$$

Section (C) Inradius and Exradius

खण्ड(C) अन्तत्रिज्या परित्रिज्या

C-1. In a ΔABC , the value of $\frac{a \cos A + b \cos B + c \cos C}{a + b + c}$ is equal to:

त्रिभुज ABC में $\frac{a \cos A + b \cos B + c \cos C}{a + b + c}$ का मान है -

- (A*) $\frac{r}{R}$ (B) $\frac{R}{2r}$ (C) $\frac{R}{r}$ (D) $\frac{2r}{R}$

Sol.
$$\frac{a \cos A + b \cos B + c \cos C}{a + b + c} = \frac{R(\sin 2A + \sin 2B + \sin 2C)}{2R(\sin A + \sin B + \sin C)}$$

$$= \frac{4 \sin A \sin B \sin C}{8 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= \frac{r}{R}$$

C-2. In a triangle ABC, if $a : b : c = 3 : 7 : 8$, then $R : r$ is equal to
यदि किसी त्रिभुज ABC में $a : b : c = 3 : 7 : 8$ हो, तो $R : r$ होगा -

- (A) $2 : 7$ (B*) $7 : 2$ (C) $3 : 7$ (D) $7 : 3$

Sol. $a = 3k ; b = 7k ; c = 8k$

$$\therefore s = 9k$$

$$\therefore \Delta = \sqrt{9k \cdot 6k \cdot 2k \cdot k} = k^2 6\sqrt{3} \quad \therefore R = \frac{abc}{4\Delta} = \frac{(3k)(7k)(8k)}{4 \times k^2 \times 6\sqrt{3}} = \frac{7k}{\sqrt{3}}$$

$$\therefore r = \frac{\Delta}{s} = \frac{k^2 6\sqrt{3}}{9k} = \frac{2k}{\sqrt{3}} \quad \therefore R : r = 7 : 2$$

C-3*. If $r_1 = 2r_2 = 3r_3$, then

यदि $r_1 = 2r_2 = 3r_3$, तो

- (A) $\frac{a}{b} = \frac{4}{5}$ (B*) $\frac{a}{b} = \frac{5}{4}$ (C) $\frac{a}{c} = \frac{3}{5}$ (D*) $\frac{a}{c} = \frac{5}{3}$

Sol. $r_1 = 2r_2 = 3r_3$

$$\Rightarrow \frac{\Delta}{s-a} = \frac{2\Delta}{s-b} = \frac{3\Delta}{s-c} \quad \Rightarrow \frac{1}{s-a} = \frac{2}{s-b} = \frac{3}{s-c}$$

(i) (ii) (iii)

From (i) and (ii) we get

$$a - b = c/3 \quad \dots(1)$$

From (i) and (iii), we get

$$2a - b = 2c \quad \dots(2)$$

From (ii) and (iii), we get

$$a - 5b = -5c \quad \dots(3)$$

let $c = k$, then from (1) and (2), we get

$$a = \frac{5k}{3} \text{ and } b = \frac{4k}{3} \quad \therefore \frac{a}{b} = \frac{5}{4} ; \frac{a}{c} = \frac{5}{3}$$





Hindi. $r_1 = 2r_2 = 3r_3$

$$\Rightarrow \frac{\Delta}{s-a} = \frac{2\Delta}{s-b} = \frac{3\Delta}{s-c} \quad \Rightarrow \quad \frac{1}{s-a} = \frac{2}{s-b} = \frac{3}{s-c}$$

(i) (ii) (iii)

(i) व (ii) से $a - b = c/3$... (1)

(i) व (iii) से $2a - b = 2c$... (2)

(ii) व (iii) से $a - 5b = -5c$... (3)

माना $c = k$ तब (1) तथा (2) को हल करने पर

$$a = \frac{5k}{3} \text{ और } b = \frac{4k}{3} \quad \therefore \frac{a}{b} = \frac{5}{4} ; \frac{a}{c} = \frac{5}{3}$$

C-4*. In a $\triangle ABC$, following relations hold good. In which case(s) the triangle is a right angled triangle?
किसी त्रिभुज ABC में निम्नलिखित सम्बन्ध सत्य है। किस स्थिति में त्रिभुज समकोण त्रिभुज है ?

(A*) $r_2 + r_3 = r_1 - r$

(B*) $a^2 + b^2 + c^2 = 8R^2$

(C*) $r_1 = s$

(D*) $2R = r_1 - r$

Sol. (A) $\frac{\Delta}{s-b} + \frac{\Delta}{s-c} = \frac{\Delta}{s-a} - \frac{\Delta}{s} \Rightarrow \frac{1}{(s-b)(s-c)} = \frac{1}{s(s-a)}$

$$\Rightarrow \tan^2 \frac{A}{2} = 1 \Rightarrow A = 90^\circ$$

(B) $4R^2 (\sin^2 A + \sin^2 B + \sin^2 C) = 8R^2$

$$1 - (\cos^2 A - \sin^2 B) + \sin^2 C = 2$$

$$1 - \cos(A+B) \cos(A-B) + 1 - \cos^2 C = 2$$

$$\cos C \cos(A-B) - \cos^2 C = 0$$

$$\cos C [\cos(A-B) - \cos C] = 0$$

$$\cos C [\cos(A-B) + \cos(A+B)] = 0$$

$$2 \cos A \cos B \cos C = 0$$

$$\Rightarrow A = 90^\circ \text{ or या } B = 90^\circ \text{ or या } C = 90^\circ$$

(C) $r_1 = s$

$$s \tan \frac{A}{2} = s \Rightarrow \tan \frac{A}{2} = 1 \Rightarrow A = 90^\circ$$

(D) $\frac{a}{\sin A} = \frac{a\Delta}{s(s-a)} \Rightarrow \frac{1}{\sin A} = \tan \frac{A}{2}$

$$\Rightarrow 2 \sin^2 \frac{A}{2} = 1$$

$$\Rightarrow 1 - \cos A = 1 \Rightarrow \cos A = 0 \Rightarrow A = 90^\circ$$

C-5. The perimeter of a triangle ABC right angled at C is 70, and the inradius is 6, then $|a - b|$ equals
कोण C पर समकोण त्रिभुज ABC का परिमाप 70 है तथा अन्तर्त्रिज्या 6 है तब $|a - b|$ बराबर है -

(A*) 1

(B) 2

(C) 8

(D) 9

Sol. We known $\Delta = sr = (70/2) \times 6 = 210$

$$\Rightarrow (1/2) ab = 210 \Rightarrow ab = 420$$

$$\text{Now } (a + b)^2 = (70 - c)^2$$

$$\Rightarrow a^2 + b^2 + 2ab = 4900 - 140c + c^2$$

$$\Rightarrow c = \frac{4900 - 2 \times 420}{140} = 29 \quad [\because a^2 + b^2 = c^2]$$

$$\Rightarrow (a - b)^2 = a^2 + b^2 - 2ab = c^2 - 2ab = 841 - 840 = 1$$

C-6. In a triangle ABC, if $\frac{a-b}{b-c} = \frac{s-a}{s-c}$, then r_1, r_2, r_3 are in:

(A*) A.P.

(B) G.P.

(C) H.P.

(D) none of these

त्रिभुज ABC में, यदि $\frac{a-b}{b-c} = \frac{s-a}{s-c}$ हो, तो r_1, r_2, r_3 हैं-

(A) समान्तर श्रेढ़ी में

(B) गुणोत्तर श्रेढ़ी में

(C) हरात्मक श्रेढ़ी में

(D) इनमें से कोई नहीं।





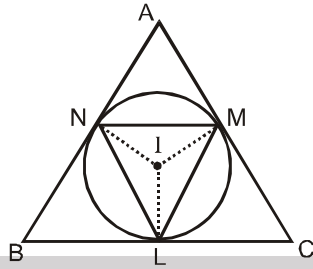
Sol. $\frac{(s-b)-(s-a)}{(s-c)-(s-b)} = \frac{s-a}{s-c} \Rightarrow \frac{\frac{\Delta}{r_2} - \frac{\Delta}{r_1}}{\frac{\Delta}{r_3} - \frac{\Delta}{r_2}} = \frac{\frac{\Delta}{r_1}}{\frac{\Delta}{r_3}} \Rightarrow \frac{(r_1 - r_2) r_2 r_3}{r_1 r_2 (r_2 - r_3)} = \frac{r_3}{r_1}$

$$\Rightarrow 2r_2 = r_1 + r_3$$

$$\Rightarrow r_1, r_2, r_3 \text{ are in A.P.}$$

$$\Rightarrow r_1, r_2, r_3 \text{ समान्तर श्रेणी में है।}$$

C-7. If the incircle of the $\triangle ABC$ touches its sides at L, M and N as shown in the figure and if x, y, z be the circumradii of the triangles MIN, NIL and LIM respectively, where I is the incentre, then the product xyz is equal to :



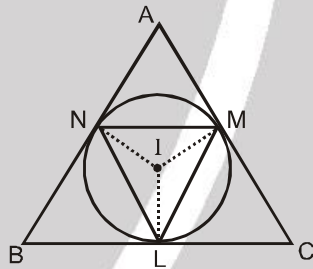
(A) Rr^2

(B) rR^2

(C*) $\frac{1}{2} Rr^2$

(D) $\frac{1}{2} rR^2$

यदि त्रिभुज ABC का अन्तःवृत्त इसकी भुजाओं को L, M एवं N पर चित्रानुसार स्पर्श करता है तथा यदि त्रिभुज MIN, NIL एवं LIM की परित्रिज्याएँ क्रमशः x, y एवं z हो, जहाँ I अन्तःकेन्द्र है, तो गुणनफल xyz का मान है -



(A) Rr^2

(B) rR^2

(C*) $\frac{1}{2} Rr^2$

(D) $\frac{1}{2} rR^2$

Sol. MINA is a cyclic quadrilateral MINA एक चक्रीय चतुर्भुज है

$$\therefore \frac{MN}{\sin A} = AI \Rightarrow MN = r \operatorname{cosec} \frac{A}{2} \sin A = 2r \cos \frac{A}{2}$$

$$IM = IN = r$$

$$\therefore x = \frac{\left(2r \cos \frac{A}{2}\right)(r)(r)}{4 \times \frac{1}{2} r \times r \sin A} = \frac{2r^3 \cos \frac{A}{2}}{2r^2 \sin A}$$

$$\frac{r \cos \frac{A}{2}}{\sin A} = \frac{r}{2 \sin \frac{A}{2}}$$

$$\text{similarly इसी प्रकार } y = \frac{r}{2 \sin \frac{B}{2}} \text{ and और } z = \frac{r}{2 \sin \frac{C}{2}}$$





$$\therefore xyz = \frac{r^3}{8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{r^3}{2 \frac{r}{R}} = \frac{1}{2} r^2 R$$

C-8. If in a $\triangle ABC$, $\frac{r}{r_1} = \frac{1}{2}$, then the value of $\tan \frac{A}{2} \left(\tan \frac{B}{2} + \tan \frac{C}{2} \right)$ is equal to :

- (A) 2 (B*) $\frac{1}{2}$ (C) 1 (D) 3

यदि त्रिभुज ABC में, $\frac{r}{r_1} = \frac{1}{2}$, हो, तो $\tan \frac{A}{2} \left(\tan \frac{B}{2} + \tan \frac{C}{2} \right)$ का मान है -

- (A) 2 (B) $\frac{1}{2}$ (C) 1 (D) 3

Sol. $\therefore \frac{r}{r_1} = \frac{1}{2} \Rightarrow \frac{4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{1}{2} \Rightarrow \tan \frac{B}{2} \tan \frac{C}{2} = \frac{1}{2} \therefore \sum \tan \frac{A}{2} \tan \frac{B}{2} = 1$

$$\Rightarrow \tan \frac{A}{2} \left(\tan \frac{B}{2} + \tan \frac{C}{2} \right) + \tan \frac{B}{2} \tan \frac{C}{2} = 1 \therefore \tan \frac{A}{2} \left(\tan \frac{B}{2} + \tan \frac{C}{2} \right) = \frac{1}{2}$$

C-9. If in a $\triangle ABC$, $\angle A = \frac{\pi}{2}$, then $\tan \frac{C}{2}$ is equal to

यदि त्रिभुज ABC में, $\angle A = \frac{\pi}{2}$ हो, तो $\tan \frac{C}{2}$ का मान है -

- (A) $\frac{a-c}{2b}$ (B) $\frac{a-b}{2c}$ (C) $\frac{a-c}{b}$ (D*) $\frac{a-b}{c}$

Sol. $\therefore a^2 = b^2 + c^2$

$$\therefore \tan C = \frac{c}{b}$$

$$\therefore \tan C = \frac{2 \tan \frac{C}{2}}{1 - \tan^2 \frac{C}{2}} = \frac{c}{b}$$

$$\frac{2t}{1-t^2} = \frac{c}{b} \text{ where (जहाँ) } t = \tan \frac{C}{2}$$

$$t^2(c) + (2b)t - c = 0$$

$$\therefore t = \frac{-2b \pm \sqrt{4b^2 + 4 \times c^2}}{2c}$$

$$t = \frac{-b \pm a}{c}$$

$$\Rightarrow t = \frac{a-b}{c} = \tan \frac{C}{2}$$

C-10. In any $\triangle ABC$, $\frac{(r_1 + r_2)(r_2 + r_3)(r_3 + r_1)}{R^2}$ is always equal to

- (A) 8 (B) 27 (C) 16 (D*) 4

किसी त्रिभुज ABC में, $\frac{(r_1 + r_2)(r_2 + r_3)(r_3 + r_1)}{R^2}$ का मान सदैव किसके बराबर होता है ?

- (A) 8 के (B) 27 के (C) 16 के (D) 4

Sol. $\therefore r_1 + r_2 = \frac{\Delta c}{(s-a)(s-b)}$





$$\begin{aligned}\therefore \Pi (r_1 + r_2) &= \frac{\Delta^3 abc}{(s-a)^2(s-b)^2(s-c)^2} = \frac{\Delta^3 (abc)s^2}{\Delta^4} \\ &= \frac{(abc)s^2}{\Delta} = \frac{4R\Delta s^2}{\Delta} = 4Rs^2 \\ \therefore \frac{\Pi (r_1 + r_2)}{Rs^2} &= 4\end{aligned}$$

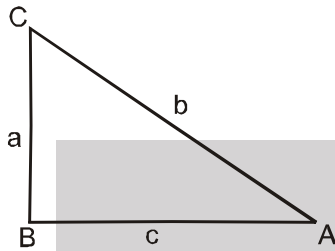
C-11*. In a triangle ABC, right angled at B, then

$$(A^*) r = \frac{AB+BC-AC}{2} \quad (B) r = \frac{AB+AC-BC}{2} \quad (C) r = \frac{AB+BC+AC}{2} \quad (D^*) R = \frac{s-r}{2}$$

त्रिभुज ABC में $\angle B$ समकोण है, तो

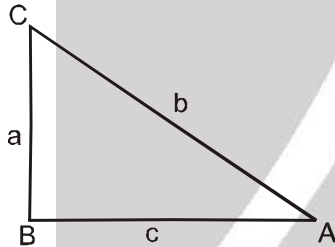
$$(A^*) r = \frac{AB+BC-AC}{2} \quad (B) r = \frac{AB+AC-BC}{2} \quad (C) r = \frac{AB+BC+AC}{2} \quad (D^*) R = \frac{s-r}{2}$$

Sol.



$$\begin{aligned}\therefore r &= (s-b) \tan \frac{B}{2} \\ r &= s-b \quad (\because B = 90^\circ) \\ \therefore r &= \frac{2s-2b}{2} = \frac{AB+BC+CA-2CA}{2} \\ \therefore r &= \frac{AB+BC-CA}{2}\end{aligned}$$

Again.



$$\begin{aligned}\therefore R &= \frac{b}{2} \\ \therefore r &= (s-b) \tan \frac{B}{2} \\ \Rightarrow r &= (s-b) \Rightarrow r = s-2R \Rightarrow R = \frac{s-r}{2}\end{aligned}$$

C-12*. With usual notations, in a ΔABC the value of $\Pi (r_1 - r)$ can be simplified as:

सामान्य संकेतानुसार त्रिभुज ABC में $\Pi (r_1 - r)$ का सरलीकृत मान है -

$$(A^*) abc \Pi \tan \frac{A}{2} \quad (B) 4rR^2 \quad (C^*) \frac{(abc)^2}{R(a+b+c)^2} \quad (D^*) 4Rr^2$$

Sol. $r_1 - r = \frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta a}{s(s-a)} = a \tan \frac{A}{2}$





$$\begin{aligned}\therefore \Pi (r_1 - r) &= abc \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} \\ &= abc \Pi \tan \frac{A}{2} \\ &= abc \frac{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{(abc)r}{4R \cdot \frac{(\sin A + \sin B + \sin C)}{4}} = \frac{(abc)r}{R \left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R} \right)} \\ &= \frac{2(abc)r}{2s} = \frac{4R\Delta r}{s} = 4Rr^2\end{aligned}$$

C-13. STATEMENT-1 : In a triangle ABC, the harmonic mean of the three exradii is three times the inradius.

STATEMENT-2 : In any triangle ABC, $r_1 + r_2 + r_3 = 4R$.

कथन -1 : एक त्रिभुज ABC में, बाह्य त्रिज्याओं का हरात्मक माध्य, अन्तः त्रिज्या का तीन गुना होता है।

कथन -2 : किसी $\triangle ABC$ में, $r_1 + r_2 + r_3 = 4R$

- (A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C*) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true
(E) Both STATEMENTS are false
- (A) कथन-1 सत्य है, कथन-2 सत्य है ; कथन-2, कथन-1 का सही स्पष्टीकरण है।
(B) कथन-1 सत्य है, कथन-2 सत्य है ; कथन-2, कथन-1 का सही स्पष्टीकरण नहीं है।
(C*) कथन-1 सत्य है, कथन-2 असत्य है।
(D) कथन-1 असत्य है, कथन-2 सत्य है।
(E) सभी कथन असत्य है।

Sol. Statement-1 :

$$\therefore \text{H.M. of the three ex-radii} = \frac{3}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}} = \frac{3\Delta}{s-a+s-b+s-c} = \frac{3\Delta}{s} = 3r$$

= 3 times the inradius

$\therefore$ statement-1 is true

Statement-2 : $\therefore$ L.H.S. = $r_1 + r_2 + r_3$

$$\begin{aligned}&= \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} \\ &= \Delta \left[\frac{(s-b)(s-c) + (s-a)(s-c) + (s-a)(s-b)}{(s-a)(s-b)(s-c)} \right] \\ &= s\Delta \left[\frac{3s^2 - 2s(a+b+c) + ab+bc+ca}{\Delta^2} \right] \\ &= \frac{s\Delta [ab+bc+ca-s^2]}{\Delta^2} \\ &= \frac{s(ab+bc+ca-s^2)}{\Delta}\end{aligned}$$

$$\therefore \text{R.H.S.} = 4R = \frac{abc}{\Delta}$$

$\therefore$ L.H.S. $\neq$ R.H.S.





∴ Statement 2 is false.

Hindi कथन-1 :

$$\begin{aligned} \therefore \text{तीन बाह्य त्रिज्याओं का हरात्मक माध्य} &= \frac{3}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}} = \frac{3\Delta}{s-a+s-b+s-c} = \frac{3\Delta}{s} = 3r \\ &= \text{अन्तःत्रिज्या का तीन गुना} \\ \therefore \text{कथन-1 सत्य है।} \end{aligned}$$

कथन-2 : ∴ बायां पक्ष = $r_1 + r_2 + r_3$

$$\begin{aligned} &= \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} \\ &= \Delta \left[\frac{(s-b)(s-c) + (s-a)(s-c) + (s-a)(s-b)}{(s-a)(s-b)(s-c)} \right] \\ &= s\Delta \left[\frac{3s^2 - 2s(a+b+c) + ab+bc+ca}{\Delta^2} \right] \\ &= \frac{s\Delta [ab+bc+ca-s^2]}{\Delta^2} \\ &= \frac{s(ab+bc+ca-s^2)}{\Delta} \end{aligned}$$

$$\begin{aligned} \therefore \text{दायां पक्ष} &= 4R = \frac{abc}{\Delta} \\ \therefore \text{बायां पक्ष} &\neq \text{दायां पक्ष} \\ \therefore \text{कथन -2 असत्य है।} \end{aligned}$$

Section (D) Miscellaneous

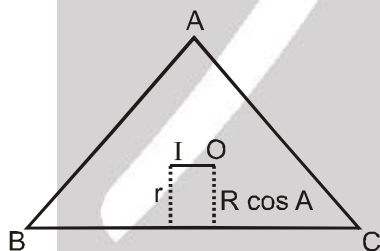
D-1. If in a triangle ABC, the line joining the circumcentre and incentre is parallel to BC, then $\cos B + \cos C$ is equal to :

- (A) 0 (B*) 1 (C) 2 (D) 1/2

यदि त्रिभुज ABC में परिकेन्द्र एवं अन्तःकेन्द्र को मिलाने वाली रेखा भुजा BC के समान्तर हो, तो $\cos B + \cos C$ का मान है—

- (A) 0 (B) 1 (C) 2 (D) 1/2

Sol.



$$\therefore R \cos A = r$$

$$R \cos A = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$\begin{aligned} \cos A &= \cos A + \cos B + \cos C - 1 \\ \cos B + \cos C &= 1 \end{aligned}$$

D-2. In a $\triangle ABC$, if $AB = 5$ cm, $BC = 13$ cm and $CA = 12$ cm, then the distance of vertex 'A' from the side BC is (in cm)





त्रिभुज ABC में यदि AB = 5 सेमी., BC = 13 सेमी. एवं CA = 12 सेमी. हो, तो शीर्ष 'A' की भुजा BC से लम्बाई है (सेमी. में) –

- (A) $\frac{25}{13}$ (B*) $\frac{60}{13}$ (C) $\frac{65}{12}$ (D) $\frac{144}{13}$

Sol. $\therefore$ required distance अभीष्ट दूरी = $\frac{2\Delta}{a}$

$$\therefore \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$a = 13; b = 12; c = 5 \Rightarrow s = 15$$

$$\therefore \Delta = \sqrt{15 \times 2 \times 3 \times 10} = 5 \times 3 \times 2 = 30$$

$$\therefore \text{required distance अभीष्ट दूरी} = \frac{2 \times 30}{13} = \frac{60}{13}$$

D-3. If AD, BE and CF are the medians of a $\triangle ABC$, then $(AD^2 + BE^2 + CF^2) : (BC^2 + CA^2 + AB^2)$ is equal to यदि किसी त्रिभुज ABC की माध्यिकाएँ AD, BE एवं CF हो, तो $(AD^2 + BE^2 + CF^2) : (BC^2 + CA^2 + AB^2)$ का मान है –

- (A) 4 : 3 (B) 3 : 2 (C*) 3 : 4 (D) 2 : 3

Sol. $\therefore AD^2 = \frac{1}{4} (2b^2 + 2c^2 - a^2),$

$$BE^2 = \frac{1}{4} (2c^2 + 2a^2 - b^2) \text{ and तथा}$$

$$CF^2 = \frac{1}{4} (2a^2 + 2b^2 - c^2) \therefore \frac{AD^2 + BE^2 + CF^2}{BC^2 + CA^2 + AB^2} = \frac{3}{4}$$

D-4. In a triangle ABC, with usual notations the length of the bisector of internal angle A is : त्रिभुज ABC में संकेतों के सामान्य प्रचलित अर्थ है, तो अन्तः कोण A के अर्धक की लम्बाई है –

- (A*) $\frac{2bc \cos \frac{A}{2}}{b+c}$ (B) $\frac{2bc \sin \frac{A}{2}}{b+c}$ (C*) $\frac{abc \operatorname{cosec} \frac{A}{2}}{2R(b+c)}$ (D*) $\frac{2\Delta}{b+c} \cdot \operatorname{cosec} \frac{A}{2}$

Sol. $\therefore \beta_a = \frac{2bc}{b+c} \cos \frac{A}{2}$

- (A) correct
(B) incorrect

$$(C) \frac{abcc \operatorname{cosec} \frac{A}{2}}{2R(b+c)} = \frac{abcc \operatorname{cosec} \frac{A}{2}}{\frac{a}{\sin A} \cdot (b+c)} = \frac{bc}{\sin \frac{A}{2}} \cdot \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{(b+c)} = \frac{2bc}{(b+c)} \cos \frac{A}{2}$$

$$(D) \therefore \frac{2\Delta}{(b+c)} \operatorname{cosec} \frac{A}{2} = \frac{bc \sin A}{(b+c)} \cdot \frac{1}{\sin \frac{A}{2}} = \frac{2bc \sin \frac{A}{2} \cos \frac{A}{2}}{(b+c)} \cdot \frac{1}{\sin \frac{A}{2}} = \frac{2bc}{b+c} \cos \frac{A}{2}$$

D-5. Let f, g, h be the lengths of the perpendiculars from the circumcentre of the $\triangle ABC$ on the sides BC, CA and AB respectively. If $\frac{a}{f} + \frac{b}{g} + \frac{c}{h} = \lambda \frac{abc}{fgh}$, then the value of ' λ ' is:

माना कि त्रिभुज ABC के परिकेन्द्र से भुजाओं BC, CA एवं AB पर डाले गये लम्बों की लम्बाईयाँ क्रमशः f, g, h हैं। यदि

$$\frac{a}{f} + \frac{b}{g} + \frac{c}{h} = \lambda \frac{abc}{fgh} \text{ हो, तो } \lambda \text{ का मान है—}$$

- (A*) $1/4$ (B) $1/2$ (C) 1 (D) 2

Sol. $f = R \cos A, g = R \cos B, h = R \cos C.$

$$\therefore \frac{a}{f} + \frac{b}{g} + \frac{c}{h} = \frac{2R \sin A}{R \cos A} + \frac{2R \sin B}{R \cos B} + \frac{2R \sin C}{R \cos C}$$



$$= 2 \left(\sum \tan A \right)$$

$$\therefore \frac{abc}{fgh} = 8 \left(\prod \tan A \right)$$

$$\therefore \frac{a}{f} + \frac{b}{g} + \frac{c}{h} = \lambda \frac{abc}{fgh} \Rightarrow 2 \sum \tan A = \lambda \cdot 8 \left(\prod \tan A \right) \Rightarrow \lambda = \frac{1}{4}$$

D-6. In an acute angled triangle ABC, AP is the altitude. Circle drawn with AP as its diameter cuts the sides AB and AC at D and E respectively, then length DE is equal to

एक न्यूनकोण त्रिभुज ABC में AP शीर्षलम्ब है। AP को व्यास मानकर खींचा गया वृत्त भुजाओं AB एवं AC को क्रमशः D एवं E पर काटता है, तो DE की लम्बाई है –

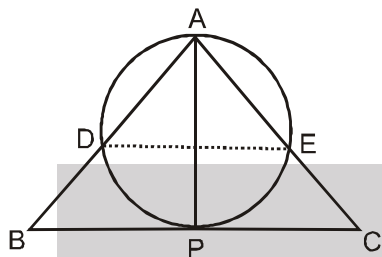
(A) $\frac{\Delta}{2R}$

(B) $\frac{\Delta}{3R}$

(C) $\frac{\Delta}{4R}$

(D*) $\frac{\Delta}{R}$

Sol.



$$\therefore \frac{DE}{\sin A} = AP \Rightarrow DE = \frac{2\Delta}{a} \sin A$$

$$= \frac{2\Delta \sin A}{2R \sin A} = \frac{\Delta}{R}$$

D-7. AA_1 , BB_1 and CC_1 are the medians of triangle ABC whose centroid is G. If points A, C_1 , G and B_1 are concyclic, then

(A) $2b^2 = a^2 + c^2$ (B) $2c^2 = a^2 + b^2$ (C*) $2a^2 = b^2 + c^2$ (D) $3a^2 = b^2 + c^2$

त्रिभुज ABC जिसका केन्द्रक G है, की माधिकाएँ AA_1 , BB_1 एवं CC_1 हैं। यदि बिन्दु A, C_1 , G एवं B_1 समचक्रीय हो, तो

(A) $2b^2 = a^2 + c^2$ (B) $2c^2 = a^2 + b^2$ (C) $2a^2 = b^2 + c^2$ (D) $3a^2 = b^2 + c^2$

Sol.

$\therefore$ A, C_1 , G and (और) B_1 are cyclic (चक्रीय हैं)

$$\therefore BC_1 \cdot BA = BG \cdot BB_1$$

$$\frac{c}{2} \cdot c = \left(\frac{2}{3} BB_1 \right) \cdot BB_1$$

$$\frac{c^2}{2} = \frac{2}{3} \times \frac{1}{4} (2c^2 + 2a^2 - b^2)$$

$$\Rightarrow c^2 + b^2 = 2a^2$$

D-8. If ' ℓ ' is the length of median from the vertex A to the side BC of a ΔABC , then

किसी ΔABC में शीर्ष A से गुजरने वाली माधिका की लम्बाई ' ℓ ' हो, तो –

(A) $4\ell^2 = b^2 + 4ac \cos B$

(B*) $4\ell^2 = a^2 + 4bc \cos A$

(C) $4\ell^2 = c^2 + 4ab \cos C$

(D) $4\ell^2 = b^2 + 2c^2 - 2a^2$

Sol.

$$\therefore \ell = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}$$

$$\therefore 4\ell^2 = 2b^2 + 2c^2 - a^2$$

$$= a^2 + 2(b^2 + c^2 - a^2)$$

$$= a^2 + 2(2bc \cos A)$$

$$4\ell^2 = a^2 + 4bc \cos A$$





D-9*. The product of the distances of the incentre from the angular points of a ΔABC is:

त्रिभुज ABC के शीर्षों से अन्तःकेन्द्र की दूरियों का गुणनफल है

- (A) $4 R^2 r$ (B*) $4 R r^2$ (C) $\frac{(a b c) R}{s}$ (D*) $\frac{(a b c) r}{s}$

Sol. Product of distances of incenter from angular points

शीर्षों से अन्तःकेन्द्र की दूरियों का गुणनफल

$$= \frac{r^3}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{r^3}{r/4R} = 4Rr^2 = \frac{abc}{\Delta} r^2 = \frac{(abc)(r)}{\frac{\Delta}{r}} = \frac{(abc)(r)}{s}$$

D-10. In a triangle ABC, $B = 60^\circ$ and $C = 45^\circ$. Let D divides BC internally in the ratio 1 : 3,

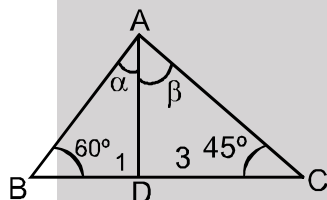
then value of $\frac{\sin \angle BAD}{\sin \angle CAD}$ is

त्रिभुज ABC में, $\angle B = 60^\circ$ एवं $\angle C = 45^\circ$ है। यदि बिन्दु D भुजा BC को 1 : 3 के अनुपात में अन्तः विभाजित करता

हो, तो $\frac{\sin \angle BAD}{\sin \angle CAD}$ का मान है -

- (A) $\sqrt{\frac{2}{3}}$ (B) $\frac{1}{\sqrt{3}}$ (C*) $\frac{1}{\sqrt{6}}$ (D) $\frac{1}{3}$

Sol.



if we apply Sine-Rule in ΔBAD , we get

ΔBAD में ज्या नियम का प्रयोग करने पर

$$\frac{BD}{\sin \alpha} = \frac{AD}{\sin 60^\circ} \quad \dots(1)$$

if we apply Sine-Rule in ΔCAD , we get. ΔCAD में ज्या नियम का प्रयोग करने पर

$$\frac{CD}{\sin \beta} = \frac{AD}{\sin 45^\circ} \quad \dots(2)$$

divide (2) by (1)

(2) में (1) का भाग देने पर

$$\frac{\sin \alpha}{\sin \beta} \times \frac{CD}{BD} = \frac{\sin 60^\circ}{\sin 45^\circ}$$

$$\frac{\sin \alpha}{\sin \beta} \times \frac{3}{1} = \frac{\sqrt{3}}{2 \times \frac{1}{\sqrt{2}}}$$

$$\frac{\sin \alpha}{\sin \beta} = \frac{1}{\sqrt{6}}$$

D-11*. In a triangle ABC, points D and E are taken on side BC such that $BD = DE = EC$. If angle $ADE = \angle AED = \theta$, then:

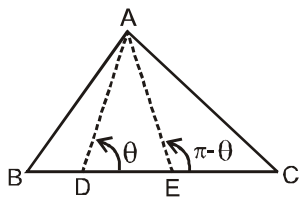
- (A*) $\tan \theta = 3 \tan B$ (B) $3 \tan \theta = \tan C$
(C*) $\frac{6 \tan \theta}{\tan^2 \theta - 9} = \tan A$ (D*) angle B = angle C

त्रिभुज ABC में, भुजा BC पर बिन्दु D एवं E इस प्रकार लिए जाते हैं कि $BD = DE = EC$. यदि $\angle ADE = \angle AED = \theta$ हो, तो -

- (A) $\tan \theta = 3 \tan B$ (B) $3 \tan \theta = \tan C$
(C) $\frac{6 \tan \theta}{\tan^2 \theta - 9} = \tan A$ (D) कोण B = कोण C



Sol.



if we apply m-n Rule in $\triangle ABE$, we get

$\triangle ABE$ में m-n प्रमेय का प्रयोग करने पर

$$(1+1) \cot \theta = 1 \cdot \cot B - 1 \cdot \cot \theta$$

$$2 \cot \theta = \cot B - \cot \theta$$

$$3 \cot \theta = \cot B$$

$$\tan \theta = 3 \tan B \quad \dots\dots\dots(1)$$

Similarly, if we apply m-n Rule in $\triangle ACD$, we get इसी प्रकार, $\triangle ACD$ में m-n प्रमेय का प्रयोग करने पर

$$(1+1) \cot (\pi - \theta) = 1 \cdot \cot \theta - 1 \cdot \cot C.$$

$$\cot C = 3 \cot \theta \Rightarrow \tan \theta = 3 \tan C \quad \dots\dots\dots(2)$$

from (1) and (2) we can say that

समीकरण (1) तथा (2) से

$$\tan B = \tan C \Rightarrow B = C$$

$$\therefore A + B + C = \pi$$

$$\therefore A = \pi - (B + C)$$

$$= \pi - 2B \quad \therefore B = C$$

$$\therefore \tan A = -\tan 2B$$

$$= -\left(\frac{2 \tan B}{1 - \tan^2 B}\right) = -\frac{\frac{2 \tan \theta}{3}}{1 - \frac{\tan^2 \theta}{9}} \Rightarrow \tan A = \frac{6 \tan \theta}{\tan^2 \theta - 9}$$

D-12. STATEMENT-1 : If R be the circumradius of a $\triangle ABC$, then circumradius of its excentral $\triangle I_1 I_2 I_3$ is $2R$.

STATEMENT-2 : If circumradius of a triangle be R, then circumradius of its pedal triangle is $\frac{R}{2}$.

- (A*) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
- (B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
- (C) STATEMENT-1 is true, STATEMENT-2 is false
- (D) STATEMENT-1 is false, STATEMENT-2 is true
- (E) Both STATEMENTS are false

कथन-1 : यदि एक $\triangle ABC$ की परित्रिज्या R है, तो बहिष्केन्द्रीय $\triangle I_1 I_2 I_3$ के बाह्य केन्द्र की परित्रिज्या $2R$ होगी।



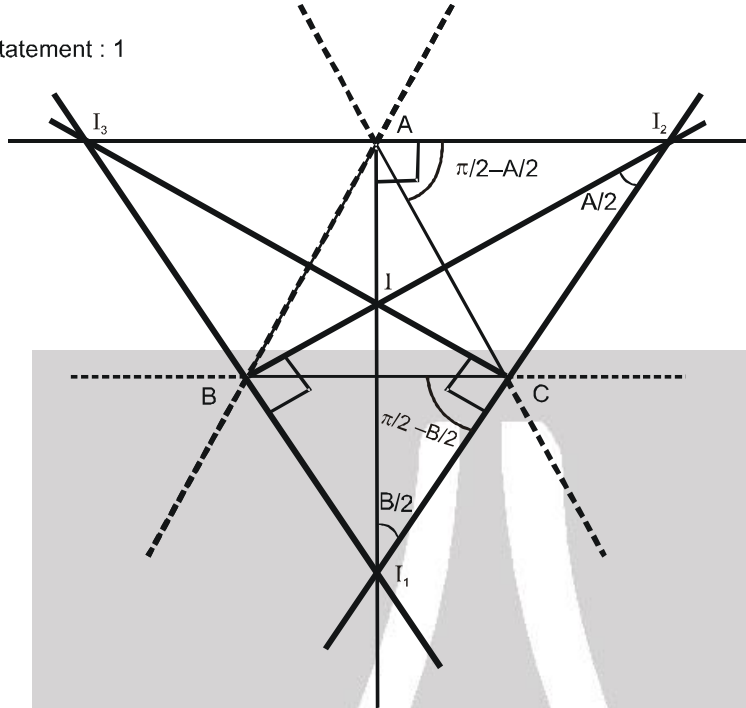


कथन-2 : यदि एक त्रिभुज की परित्रिज्या R है, तो इसके पदिक त्रिभुज की परित्रिज्या $\frac{R}{2}$ होगी।

- (A*) कथन-1 सत्य है, कथन-2 सत्य है ; कथन-2, कथन-1 का सही स्पष्टीकरण है।
 (B) कथन-1 सत्य है, कथन-2 सत्य है ; कथन-2, कथन-1 का सही स्पष्टीकरण नहीं है।
 (C) कथन-1 सत्य है, कथन-2 असत्य है।
 (D) कथन-1 असत्य है, कथन-2 सत्य है।
 (E) सभी कथन असत्य है।

Sol.

Statement : 1



$I_1 I_2 = 4R \cos \frac{C}{2}$ if we apply Sine-Rule in $\Delta I_1 I_2 I_3$, then $\Delta I_1 I_2 I_3$ में ज्या नियम का प्रयोग करने पर—

$$2R_{\text{ex}} = \frac{I_1 I_2}{\sin\left(\frac{A}{2} + \frac{B}{2}\right)} = \frac{4R \cos \frac{C}{2}}{\sin\left(\frac{A+B}{2}\right)}$$

$$= \frac{4R \cos \frac{C}{2}}{\cos \frac{C}{2}}$$

$$2R_{\text{ex}} = 4R \quad R_{\text{ex}} = 2R$$

$\therefore \Delta ABC$ is pedal triangle of $\Delta I_1 I_2 I_3$

ΔABC , त्रिभुज $\Delta I_1 I_2 I_3$ का पदिक त्रिभुज है।

$\therefore$ statement - 1 and statement - 2 both are correct and statement - 2 also explains Statement - 1
 कथन - 1 तथा कथन - 2 दोनों सत्य है तथा कथन - 2, कथन- 1 सही व्याख्या भी करता है।

PART-II (COMPREHENSION)

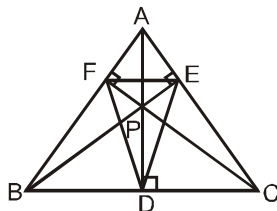
अनुच्छेद (COMPREHENSION)

Comprehension # 1 (Q. No. 1 to 4)



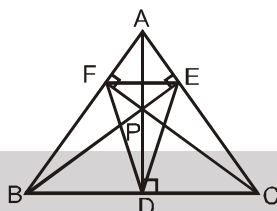
The triangle DEF which is formed by joining the feet of the altitudes of triangle ABC is called the Pedal Triangle.

Answer The Following Questions :



अनुच्छेद # 1

त्रिभुज ABC शीर्ष लम्बों पादों को मिलाने से बना त्रिभुज DEF दिए गए त्रिभुज का पदिक त्रिभुज कहलाता है, निम्न प्रश्नों के उत्तर दीजिए



1. ✖

Angle of triangle DEF are

(A*) $\pi - 2A, \pi - 2B$ and $\pi - 2C$

(C) $\pi - A, \pi - B$ and $\pi - C$

त्रिभुज DEF के कोण है

(A) $\pi - 2A, \pi - 2B$ और $\pi - 2C$

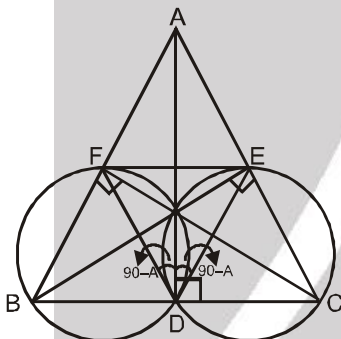
(C) $\pi - A, \pi - B$ और $\pi - C$

(B) $\pi + 2A, \pi + 2B$ and $\pi + 2C$

(D) $2\pi - A, 2\pi - B$ and $2\pi - C$

(B) $\pi + 2A, \pi + 2B$ और $\pi + 2C$

(D) $2\pi - A, 2\pi - B$ और $2\pi - C$



Sol.

$$\angle EDF = 90 - A + 90 - A \\ = 180 - A$$

2*. ✖

Sides of triangle DEF are

त्रिभुज DEF की भुजाएँ है—

(A) $b \cos A, a \cos B, c \cos C$

(C*) $R \sin 2A, R \sin 2B, R \sin 2C$

(B*) $a \cos A, b \cos B, c \cos C$

(D) $a \cot A, b \cot B, c \cot C$

Sol.

$$\triangle AEF : AF = b \cos A, AE = c \cos A$$

$$\therefore \cos A = \frac{b^2 \cos^2 A + C^2 \cos^2 A - EF^2}{2bc \cos A \cdot c \cos A}$$

$$\Rightarrow (EF)^2 = (b^2 + c^2 - 2bc \cos A) \cos^2 A$$

$$(EF)^2 = a^2 \cos^2 A$$

$$EF = a \cos A$$

3. ✖

Circumradii of the triangle PBC, PCA and PAB are respectively

त्रिभुज PBC, PCA तथा PAB की परित्रिज्याएँ क्रमशः है—

(A*) R, R, R

(B) $2R, 2R, 2R$

(C) $R/2, R/2, R/2$

(D) $3R, 3R, 3R$





Sol. Circumradius of the triangle PBC = $\frac{BC}{2 \sin(B+C)}$

$$= \frac{a}{2 \sin(\pi - A)} = \frac{a}{2 \sin A} = R$$

4* Which of the following is/are correct

(A*) $\frac{\text{Perimeter of } \triangle DEF}{\text{Perimeter of } \triangle ABC} = \frac{r}{R}$

(B*) Area of $\triangle DEF = 2 \Delta \cos A \cos B \cos C$

(C*) Area of $\triangle AEF = \Delta \cos^2 A$
निम्न में से कौनसा सही है—

(D*) Circum-radius of $\triangle DEF =$

(A) $\frac{\text{त्रिभुज DEF का परिमाप}}{\text{त्रिभुज ABC का परिमाप}} = \frac{r}{R}$

(B) त्रिभुज DEF का क्षेत्रफल = $2 \Delta \cos A \cos B \cos C$

(C) त्रिभुज AEF का क्षेत्रफल = $\Delta \cos^2 A$

(D) त्रिभुज DEF की परित्रिज्या = $\frac{R}{2}$

Sol. $\therefore FE = a \cos A = R \sin 2A$

$DE = c \cos C = R \sin 2C$

$FD = b \cos B = R \sin 2B$

(A) $= \frac{R(\sum \sin 2A)}{a+b+c}$

$$= \frac{R(4 \sin A \sin B \sin C)}{2R \left(4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}\right)} = \frac{8 \left(\prod \sin \frac{A}{2}\right) \left(\prod \cos \frac{A}{2}\right)}{2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{r}{R}$$

(B) $\therefore$ Area of $\triangle DEF$ का क्षेत्रफल

$$= \frac{1}{2} FD \times DE \sin(\pi - 2A) = \frac{1}{2} b \cos B \cdot c \cos C \cdot \sin 2A$$

$$= \frac{1}{2} bc \cos B \cdot \cos C \cdot 2 \sin A \cdot \cos A = 2 \left(\frac{1}{2} bc \sin A\right) \cos A \cdot \cos B \cdot \cos C$$

$$= 2 \Delta \cos A \cdot \cos B \cdot \cos C$$

(C) Area of $\triangle AEF$ का क्षेत्रफल = $\frac{1}{2} AE \times AF \sin A$

$$= \frac{1}{2} (c \cos A) (b \cos A) \sin A = \left(\frac{1}{2} bc \sin A\right) \cos^2 A = \Delta \cos^2 A$$

(D) $R_{DEF} = \frac{FE \times DE \times FD}{4 \Delta_{DEF}} = \frac{abc \cos A \cos B \cos C}{4 \times 2 \Delta \cos A \cos B \cos C} = \frac{abc}{8 \Delta} = \frac{4R \Delta}{8 \Delta} = \frac{R}{2}$

Comprehension # 2 (Q. 5 to 8)

अनुच्छेद # 2

The triangle formed by joining the three excentres I_1 , I_2 and I_3 of $\triangle ABC$ is called the excentral or excentric triangle and in this case internal angle bisector of triangle ABC are the altitudes of triangles $I_1 I_2 I_3$

त्रिभुज ABC के तीन बहिष्केन्द्रो I_1 , I_2 तथा I_3 को मिलाने से बना त्रिभुज बहिष्केन्द्रीय त्रिभुज कहलाता है तथा इस स्थिति में $\triangle ABC$ के आन्तरिक कोण अर्द्धक त्रिभुज $I_1 I_2 I_3$ के शीर्ष लम्ब है।

5* Incentre I of $\triangle ABC$ is the of the excentral $\triangle I_1 I_2 I_3$.

(A) Circumcentre (B*) Orthocentre (C) Centroid

(D) None of these

त्रिभुज ABC का अन्तःकेन्द्र I, बहिष्केन्द्र त्रिभुज $I_1 I_2 I_3$ का है—

(A) परिकेन्द्र (B) लम्बकेन्द्रक (C) केन्द्रक

(D) इनमें से कोई नहीं



Sol. Clearly

6. Angles of the $\Delta I_1 I_2 I_3$ are

(A*) $\frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2}$ and $\frac{\pi}{2} - \frac{C}{2}$

(B) $\frac{\pi}{2} + \frac{A}{2}, \frac{\pi}{2} + \frac{B}{2}$ and $\frac{\pi}{2} + \frac{C}{2}$

(C) $\frac{\pi}{2} - A, \frac{\pi}{2} - B$ and $\frac{\pi}{2} - C$

(D) None of these

त्रिभुज $I_1 I_2 I_3$ के कोण हैं—

(A*) $\frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2}$ तथा $\frac{\pi}{2} - \frac{C}{2}$

(B) $\frac{\pi}{2} + \frac{A}{2}, \frac{\pi}{2} + \frac{B}{2}$ तथा $\frac{\pi}{2} + \frac{C}{2}$

(C) $\frac{\pi}{2} - A, \frac{\pi}{2} - B$ तथा $\frac{\pi}{2} - C$

(D) इनमें से कोई नहीं

Sol. Let $\angle I_3 I_1 I_2 = \theta$

Then angle of pedal triangle पदिक त्रिभुज के कोण $= \pi - 2\theta = A$

$$\theta = \frac{\pi}{2} - \frac{A}{2}$$

7. Sides of the $\Delta I_1 I_2 I_3$ are

(A) $R \cos \frac{A}{2}, R \cos \frac{B}{2}$ तथा $R \cos \frac{C}{2}$

(B*) $4R \cos \frac{A}{2}, 4R \cos \frac{B}{2}$ तथा $4R \cos \frac{C}{2}$

(C) $2R \cos \frac{A}{2}, 2R \cos \frac{B}{2}$ तथा $2R \cos \frac{C}{2}$

(D) None of these

त्रिभुज $I_1 I_2 I_3$ की भुजाएँ हैं—

(A) $R \cos \frac{A}{2}, R \cos \frac{B}{2}$ and $R \cos \frac{C}{2}$

(B*) $4R \cos \frac{A}{2}, 4R \cos \frac{B}{2}$ and $4R \cos \frac{C}{2}$

(C) $2R \cos \frac{A}{2}, 2R \cos \frac{B}{2}$ and $2R \cos \frac{C}{2}$

(D) इनमें से कोई नहीं

Sol. Side of pedal triangle पदिक त्रिभुज की भुजाएँ $= I_2 I_3 \cos \theta = BC$

$$I_2 I_3 = \frac{a}{\cos \left(\frac{\pi}{2} - \frac{A}{2} \right)}$$

$$I_2 I_3 = 4R \cos \left(\frac{A}{2} \right)$$

8. Value of $I_1 I_2^2 + I_2 I_3^2 = I_2^2 + I_3 I_1^2 = I_3^2 + I_1 I_2^2 =$
 $I_1 I_2^2 + I_2 I_3^2 = I_2^2 + I_3 I_1^2 = I_3^2 + I_1 I_2^2 =$ का मान है—

(A) $4R^2$

(B*) $16R^2$

(C) $32R^2$

(D) $64R^2$

Sol $I I_1 = 4R \sin \frac{A}{2}$

$$I_2 I_3 = 4R \cos \frac{A}{2}$$

$$\therefore I_1^2 + I_2 I_3^2 = 16R^2$$

PART-III (MATCH THE COLUMN)

भाग— III (कॉलम को सुमेलित कीजिए (MATCH THE COLUMN))

1. Match the column

Column— I

Column—II



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ADVSOT - 39



- (A) In a $\triangle ABC$, $2B = A + C$ and $b^2 = ac$. (p) 8
Then the value of $\frac{a^2(a+b+c)}{3abc}$ is equal to
- (B) In any right angled triangle ABC , the value of $\frac{a^2+b^2+c^2}{R^2}$ (q) 1
is always equal to (where R is the circumradius of $\triangle ABC$)
- (C) In a $\triangle ABC$ if $a = 2$, $bc = 9$, then the value of $2R\Delta$ is equal to (r) 5
- (D) In a $\triangle ABC$, $a = 5$, $b = 3$ and $c = 7$, then the value of (s) 9
 $3 \cos C + 7 \cos B$ is equal to

Ans. (A) $\rightarrow$ (q), (B) $\rightarrow$ (p), (C) $\rightarrow$ (s), (D) $\rightarrow$ (r)
मिलान कीजिए

स्तम्भ - I

स्तम्भ - II

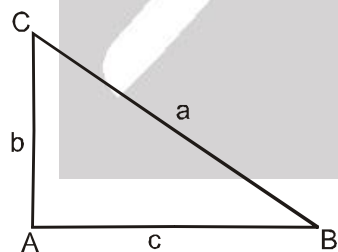
- (A) किसी त्रिभुज ABC में, $2B = A + C$ और $b^2 = ac$ हो, तो (p) 8
 $\frac{a^2(a+b+c)}{3abc}$ का मान होगा—
- (B) किसी समकोण त्रिभुज ABC में, $\frac{a^2+b^2+c^2}{R^2}$ का मान हमेशा होता है— (q) 1
(जहाँ R , $\triangle ABC$ की परित्रिज्या है)
- (C) किसी $\triangle ABC$ में, यदि $a = 2$, $bc = 9$ हो, तो $2R\Delta$ का मान है— (r) 5
- (D) किसी $\triangle ABC$ में, $a = 5$, $b = 3$ और $c = 7$ है, तब (s) 9
 $3 \cos C + 7 \cos B$ का मान होगा—

Ans. (A) $\rightarrow$ (q), (B) $\rightarrow$ (p), (C) $\rightarrow$ (s), (D) $\rightarrow$ (r)

Sol.

$$\begin{aligned} \text{(A)} \quad & \because 2B = A + C \Rightarrow B = \frac{\pi}{3} \text{ and } A + C = \frac{2\pi}{3} \\ & \because b^2 = ac \Rightarrow \sin^2 B = \sin A \cdot \sin C \Rightarrow \sin A \sin C = \frac{3}{4} \\ & \Rightarrow \cos(A - C) - \cos(A + C) = \frac{3}{2} \quad \because A + C = \frac{2\pi}{3} \\ & \Rightarrow \cos(A - C) = 1 \Rightarrow A = C = \frac{\pi}{3} = B \Rightarrow a = b = c \quad \therefore \frac{a^2(a+b+c)}{3abc} = 1 \end{aligned}$$

$$\text{(B)} \quad \because a^2 = b^2 + c^2 \text{ and } \text{और } 2R = a \quad \therefore \frac{a^2+b^2+c^2}{R^2} = \frac{2a^2}{R^2} = 8$$



$$\text{(C)} \quad \because \Delta = \frac{1}{2} bc \sin A \Rightarrow \Delta = \frac{1}{2} \cdot 9 \cdot \sin A = \frac{9}{2} \times \frac{a}{2R} \quad \because a = 2$$

$$\therefore 2R\Delta = 9$$

$$\text{(D)} \quad \because a = 5, b = 3 \text{ and } \text{और } c = 7$$

and because we know that और हम जानते हैं कि



$$b \cos C + c \cos B = a$$

$$\therefore 3 \cos C + 7 \cos B = 5$$

2. Match the column

Column – I

(A) In a $\triangle ABC$, $a = 4$, $b = 3$ and the medians AA_1 and BB_1 are mutually perpendicular, then square of area of the $\triangle ABC$ is equal to

(B) In any $\triangle ABC$, minimum value of $\frac{r_1 r_2 r_3}{r^3}$ is equal to

(C) In a $\triangle ABC$, $a = 5$, $b = 4$ and $\tan \frac{C}{2} = \sqrt{\frac{7}{9}}$, then side 'c' is equal to

(D) In a $\triangle ABC$, $2a^2 + 4b^2 + c^2 = 4ab + 2ac$, then value of $(8 \cos B)$ is equal to

Ans. (A) $\rightarrow$ (s), (B) $\rightarrow$ (p), (C) $\rightarrow$ (r), (D) $\rightarrow$ (q)

Column – II

(p) 27

(q) 7

(r) 6

(s) 11

मिलान कीजिए

स्तम्भ – I

(A) किसी त्रिभुज $\triangle ABC$ में, $a = 4$, $b = 3$ तथा माध्यिकाएं AA_1 और BB_1 परस्पर लम्बवत् है तब $\triangle ABC$ के क्षेत्रफल के वर्ग का मान होगा –

(B) किसी त्रिभुज ABC में, $\frac{r_1 r_2 r_3}{r^3}$ का न्यूनतम मान है–

(C) यदि त्रिभुज $\triangle ABC$ में, $a = 5$, $b = 4$ और $\tan \frac{C}{2} = \sqrt{\frac{7}{9}}$ हो, तो भुजा c का मान होगा –

(D) त्रिभुज $\triangle ABC$ में, $2a^2 + 4b^2 + c^2 = 4ab + 2ac$, तब $8 \cos B$ का मान होगा–

Ans. (A) $\rightarrow$ (s), (B) $\rightarrow$ (p), (C) $\rightarrow$ (r), (D) $\rightarrow$ (q)

स्तम्भ – II

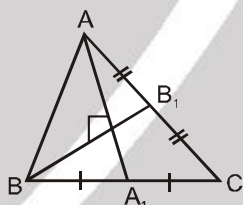
(p) 27

(q) 7

(r) 6

(s) 11

Sol.



Match the column स्तम्भ मिलान कीजिए

(A) AA_1 and BB_1 are perpendicular
 AA_1 तथा BB_1 परस्पर लम्बवत् है,

$$\therefore a^2 + b^2 = 5c^2$$

$$\therefore c^2 = \frac{a^2 + b^2}{5} = 5 \quad \Rightarrow \quad c = \sqrt{5}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{16 + 9 - 5}{2 \times 4 \times 3} = \frac{5}{6}$$

$$\sin C = \frac{\sqrt{11}}{6}$$



$$\therefore \Delta = \frac{1}{2} ab \sin C = \sqrt{11}$$

$$\therefore \Delta^2 = 11$$

$$(B) \quad \therefore G.M. \geq H.M. \quad \therefore \text{गुणोत्तर माध्य} \geq \text{समान्तर माध्य}$$

$$(r_1 r_2 r_3)^{1/3} \geq \frac{3}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}}$$

$$\Rightarrow (r_1 r_2 r_3)^{1/3} \geq 3r$$

$$\Rightarrow \frac{r_1 r_2 r_3}{r^3} \geq 27$$

$$(C) \quad \tan^2 \frac{C}{2} = \frac{(s-a)(s-b)}{s(s-c)} \quad \therefore a = 5, b = 4 \quad 2s = 9 + c$$

$$= \frac{(9+c-10)(9+c-8)}{(9+c)(9-c)} = \frac{c^2 - 1}{81 - c^2}$$

$$\Rightarrow \frac{7}{9} = \frac{c^2 - 1}{81 - c^2} \quad \Rightarrow c^2 = 36 \quad \Rightarrow c = 6$$

$$(D) \quad \therefore 2a^2 + 4b^2 + c^2 = 4ab + 2ac.$$

$$\Rightarrow (a - 2b)^2 + (a - c)^2 = 0$$

$$\Rightarrow a = 2b = c$$

$$\therefore \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{7}{8}$$

$$\therefore 8 \cos B = 7$$

Exercise-3

Marked Questions may have for Revision Questions.

चिन्हित प्रश्न दोहराने योग्य प्रश्न है।

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

भाग - I : JEE (ADVANCED) / IIT-JEE (पिछले वर्षों) के प्रश्न

* Marked Questions may have more than one correct option.

* चिन्हित प्रश्न एक से अधिक सही विकल्प वाले प्रश्न है -

1. If the angle A, B and C of a triangle are in arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c} \sin 2C + \frac{c}{a} \sin 2A$ is

यदि किसी त्रिभुज के कोण A, B एवं C समान्तर श्रेढी में हैं तथा कोणों A, B एवं C की सम्मुख भुजाओं की लम्बाईयाँ

क्रमशः a, b तथा c हैं, तो व्यंजक $\frac{a}{c} \sin 2C + \frac{c}{a} \sin 2A$ का मान है— [IIT-JEE 2010, Paper-1, (3, -1), 84]

$$(A) \frac{1}{2}$$

$$(B) \frac{\sqrt{3}}{2}$$

$$(C) 1$$

$$(D^*) \sqrt{3}$$

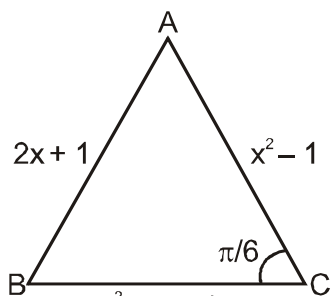
$$\text{Sol.} \quad \frac{a}{c} \sin 2C + \frac{c}{a} \sin 2A = \frac{2}{2R} (a \cos C + c \cos A) = \frac{b}{R} = 2 \sin B = 2 \sin 60^\circ = \sqrt{3}$$



2. Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are)
- माना कि ABC एक त्रिभुज है जिसमें $\angle ACB = \frac{\pi}{6}$ तथा A, B तथा C की सम्मुख भुजाओं की लम्बाइयाँ क्रमशः a, b तथा c हैं। x के वह मान जिनके लिए $a = x^2 + x + 1$, $b = x^2 - 1$ एवं $c = 2x + 1$ हों, निम्न हैं
- (A) $-(2 + \sqrt{3})$ (B*) $1 + \sqrt{3}$ (C) $2 + \sqrt{3}$ (D) $4\sqrt{3}$

[IIT-JEE 2010, Paper-1, (3, 0), 84]

Sol.



$$\cos \frac{\pi}{6} = \frac{x^2 + x + 1}{(x^2 - 1)^2 + (x^2 + x + 1)^2 - (2x + 1)^2}$$

$$\frac{\sqrt{3}}{2} = \frac{(x^2 - 1)^2 + (x^2 + 3x + 2)(x^2 - x)}{2(x^2 + x + 1)(x^2 - 1)}$$

$$\frac{\sqrt{3}}{2} = \frac{(x^2 - 1)^2 + (x + 1)(x + 2)x(x - 1)}{2(x^2 + x + 1)(x^2 - 1)}$$

$$\Rightarrow \sqrt{3} = \frac{x^2 - 1 + x(x + 2)}{x^2 + x + 1} \Rightarrow \sqrt{3}(x^2 + x + 1) = 2x^2 + 2x - 1$$

$$\Rightarrow (\sqrt{3} - 2)x^2 + (\sqrt{3} - 2)x + (\sqrt{3} + 1) = 0$$

on solving हल करने पर

$$x^2 + x - (3\sqrt{3} + 5) = 0 \quad \text{we get}$$

$$x = \sqrt{3} + 1, -(2 + \sqrt{3})$$

$\therefore$ At $x = -(2 + \sqrt{3})$, Side c becomes negative. पर भुजा c ऋणात्मक होती है

$\therefore x = \sqrt{3} + 1$

3. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. Suppose $a = 6$, $b = 10$ and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to [IIT-JEE 2010, Paper-2, (3, 0), 79]
- दिया है कि किसी त्रिभुज ABC के शीर्षों A, B एवं C की सम्मुख भुजाओं की लम्बाइयाँ क्रमशः a, b एवं c हैं। माना कि $a = 6$, $b = 10$ तथा त्रिभुज का क्षेत्रफल $15\sqrt{3}$ है। यदि $\angle ACB$ अधिक कोण (obtuse angle) है तथा त्रिभुज के अन्तःवृत्त की त्रिज्या r है, तो r^2 का मान ज्ञात कीजिए।

Ans.

3

Sol. Area of triangle त्रिभुज का क्षेत्रफल $= \frac{1}{2} ab \sin C = 15\sqrt{3}$

$$\Rightarrow \frac{1}{2} \cdot 6 \cdot 10 \sin C = 15\sqrt{3} \Rightarrow \sin C = \frac{\sqrt{3}}{2}$$

$$\Rightarrow C = \frac{2\pi}{3} \quad (C \text{ is obtuse angle अधिक कोण})$$

Now तथा $\cos C = \frac{a^2 + b^2 - c^2}{2ab} \Rightarrow -\frac{1}{2} = \frac{36 + 100 - c^2}{2 \cdot 6 \cdot 10} \Rightarrow c = 14$





$$\therefore r = \frac{\Delta}{s} = \frac{15\sqrt{3}}{6+10+14} = \sqrt{3} \Rightarrow r^2 = 3$$

4. Let PQR be a triangle of area Δ with $a = 2$, $b = \frac{7}{2}$ and $c = \frac{5}{2}$, where a , b and c are the lengths of the sides of the triangle opposite to the angles at P , Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals (SOT)

त्रिभुज PQR का क्षेत्रफल Δ है जिसके लिए $a = 2$, $b = \frac{7}{2}$ और $c = \frac{5}{2}$ है, जहाँ a , b और c क्रमशः कोण P , Q और R की सम्मुख भुजाओं की लम्बाइयाँ हैं। तब $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ का मान निम्न है— [IIT-JEE 2012, Paper-2, (3, - 1), 66]

- (A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$ (C*) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$

Sol. Ans. (C)

$$a = 2 = QR$$

$$b = \frac{7}{2} = PR$$

$$c = \frac{5}{2} = PQ$$

$$s = \frac{a+b+c}{2} = \frac{8}{4} = 4$$

$$\frac{2\sin P - 2\sin P \cos P}{2\sin P + 2\sin P \cos P} = \frac{2\sin P(1 - \cos P)}{2\sin P(1 + \cos P)} = \frac{1 - \cos P}{1 + \cos P} = \frac{2\sin^2 \frac{P}{2}}{2\cos^2 \frac{P}{2}} = \tan^2 \frac{P}{2}$$

$$= \frac{(s-b)(s-c)}{s(s-a)} = \frac{(s-b)^2(s-c)^2}{\Delta^2} = \frac{\left(4 - \frac{7}{2}\right)^2 \left(4 - \frac{5}{2}\right)^2}{\Delta^2} = \left(\frac{3}{4\Delta}\right)^2$$

Hindi.

$$a = 2 = QR$$

$$b = \frac{7}{2} = PR$$

$$c = \frac{5}{2} = PQ$$

$$s = \frac{a+b+c}{2} = \frac{8}{4} = 4$$

$$\frac{2\sin P - 2\sin P \cos P}{2\sin P + 2\sin P \cos P} = \frac{2\sin P(1 - \cos P)}{2\sin P(1 + \cos P)} = \frac{1 - \cos P}{1 + \cos P} = \frac{2\sin^2 \frac{P}{2}}{2\cos^2 \frac{P}{2}} = \tan^2 \frac{P}{2}$$

$$= \frac{(s-b)(s-c)}{s(s-a)} = \frac{(s-b)^2(s-c)^2}{\Delta^2} = \frac{\left(4 - \frac{7}{2}\right)^2 \left(4 - \frac{5}{2}\right)^2}{\Delta^2} = \left(\frac{3}{4\Delta}\right)^2$$

- 5.* In a triangle PQR, P is the largest angle and $\cos P = \frac{1}{3}$. Further the incircle of the triangle touches the sides PQ , QR and RP at N , L and M respectively, such that the lengths of PN , QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are)



त्रिभुज PQR में, P वृहत्तम कोण है तथा $\cos P = \frac{1}{3}$ । इसके अतिरिक्त त्रिभुज का अन्तःवृत्त भुजाओं PQ, QR तथा RP को क्रमशः N, L तथा M पर इस तरह स्पर्श करता है कि PN, QL तथा RM की लम्बाईयाँ क्रमागत सम पूर्ण संख्याएँ हैं। तब त्रिभुज की भुजा (भुजाओं) की सम्भावित लम्बाई (लम्बाईयाँ) है (हैं) **[JEE (Advanced) 2013, Paper-2, (3, -1)/60]**

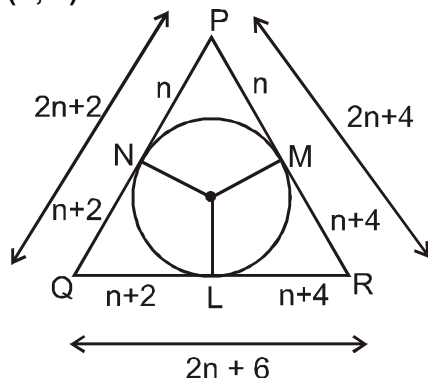
(A) 16

(B*) 18

(C) 24

(D*) 22

Sol.



$$\begin{aligned}\cos P &= \frac{(2n+2)^2 + (2n+4)^2 - (2n+6)^2}{2(2n+2)(2n+4)} = \frac{1}{3} \\ \Rightarrow \frac{4n^2 - 16}{8(n+1)(n+2)} &= \frac{1}{3} \\ = \frac{n^2 - 4}{2(n+1)(n+2)} &= \frac{1}{3} \quad \Rightarrow \quad \frac{n-2}{2(n+1)} = \frac{1}{3} \\ = 3n - 6 &= 2n + 2 \\ \Rightarrow n &= 8 \\ \Rightarrow 2n + 2 &= 18 \\ \Rightarrow 2n + 4 &= 22 \\ \Rightarrow 2n + 6 &= 24\end{aligned}$$

6. In a triangle the sum of two sides is x and the product of the same two sides is y . If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is

[JEE (Advanced) 2014, Paper-2, (3, -1)/60]

(A) $\frac{3y}{2x(x+c)}$

(B) $\frac{3y}{2c(x+c)}$

(C) $\frac{3y}{4x(x+c)}$

(D) $\frac{3y}{4c(x+c)}$

एक त्रिभुज की दो भुजाओं का योग x है तथा उन्हीं भुजाओं का गुणनफल y है। यदि $x^2 - c^2 = y$, जहाँ c त्रिभुज की तीसरी भुजा है, तब त्रिभुज की अंतःत्रिज्या (in-radius) एवं परिवृत्त-त्रिज्या (circum-radius) का अनुपात (ratio) है—

[JEE (Advanced) 2014, Paper-2, (3, -1)/60]

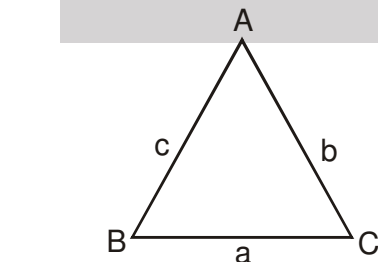
(A) $\frac{3y}{2x(x+c)}$

(B) $\frac{3y}{2c(x+c)}$

(C) $\frac{3y}{4x(x+c)}$

(D) $\frac{3y}{4c(x+c)}$

Ans.
Sol.



$a + b = x$
 $ab = y$



$$\begin{aligned}x^2 - c^2 &= y \\(a+b)^2 - c^2 &= ab \\a^2 + b^2 + ab &= c^2\end{aligned}$$

$$\begin{aligned}a^2 + b^2 - c^2 &= -ab \\ \frac{a^2 + b^2 - c^2}{2ab} &= \frac{-1}{2}\end{aligned}$$

$$\cos C = \frac{-1}{2}$$

$$C = \frac{2\pi}{3}$$

$$\begin{aligned}\frac{r}{R} &= \frac{\Delta \times 4\Delta}{s \times abc} = \frac{4 \times \frac{1}{4} a^2 b^2 \sin^2 C}{(a+b+c)abc} = \frac{3ab}{4c(x+c)} \\ &= \frac{3y}{4c(x+c)}\end{aligned}$$

- 7*. In a triangle XYZ, let x, y, z be the lengths of sides opposite to the angles X, Y, Z , respectively, and $2s = x + y + z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then

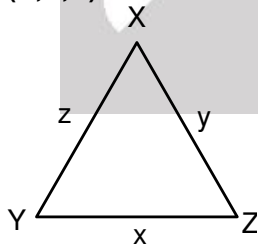
[JEE (Advanced) 2016, Paper-1, (4, -2)/62]

- (A) area of the triangle XYZ is $6\sqrt{6}$
 (B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$
 (C) $\sin \frac{X}{2} \sin \frac{Y}{2} \sin \frac{Z}{2} = \frac{4}{35}$
 (D) $\sin^2 \left(\frac{X+Y}{2} \right) = \frac{3}{5}$

माना कि त्रिभुज XYZ में कोणों X, Y, Z के सामने की भुजाओं की लम्बाइयाँ क्रमाः x, y, z है और $2s = x + y + z$ है। यदि $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$, और त्रिभुज XYZ के अंतर्वृत्त (incircle) का क्षेत्रफल $\frac{8\pi}{3}$ है, तब

- (A) त्रिभुज XYZ का क्षेत्रफल $6\sqrt{6}$ है
 (B) त्रिभुज XYZ के परिवृत्त (circumcircle) की त्रिज्या $\frac{35}{6}\sqrt{6}$ है
 (C) $\sin \frac{X}{2} \sin \frac{Y}{2} \sin \frac{Z}{2} = \frac{4}{35}$
 (D) $\sin^2 \left(\frac{X+Y}{2} \right) = \frac{3}{5}$

Ans. (A,C,D)



Sol.

$$\begin{aligned}2S &= x + y + z \quad \Rightarrow \quad \frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} = \lambda \\ S - x &= 4\lambda\end{aligned}$$



$$S - y = 3\lambda$$

$$S - z = 2\lambda$$

$$S = 9\lambda$$

Adding all we get सभी को जोड़ने पर, हम पाते हैं—

$$S = 9\lambda, x = 5\lambda, y = 6\lambda, z = 7\lambda$$

$$\pi r^2 = \frac{8\pi}{3} \Rightarrow r^2 = \frac{8}{3}$$

$$\Delta = \sqrt{S(S-x)(S-y)(S-z)} \Rightarrow \Delta = \sqrt{9\lambda \cdot 4\lambda \cdot 3\lambda \cdot 2\lambda} = 6\sqrt{6}\lambda^2$$

$$R = \frac{xyz}{4\Delta} = \frac{5\lambda \cdot 6\lambda \cdot 7\lambda}{4 \cdot 6\sqrt{6}\lambda^2} = \frac{35}{4\sqrt{6}}\lambda \Rightarrow r^2 = \frac{8}{3} = \frac{\Delta^2}{S^2} = \frac{216\lambda^4}{81\lambda^2} = \frac{24}{9}\lambda^2 = \frac{8}{3}\lambda^2 = \frac{8}{3}$$

we get हम पाते हैं $\lambda = 1$

$$(A) \Delta = 6\sqrt{6} \quad (B) R = \frac{35}{4\sqrt{6}}\lambda = \frac{35}{4\sqrt{6}}$$

$$(C) r = 4R \sin \frac{X}{2} \sin \frac{Y}{2} \sin \frac{Z}{2} \Rightarrow \frac{2\sqrt{2}}{\sqrt{3}} = 4 \cdot \frac{35}{4\sqrt{6}} \cdot \sin \frac{x}{2} \sin \frac{y}{2} \sin \frac{z}{2}$$

$$\frac{4}{35} = \sin \frac{x}{2} \sin \frac{y}{2} \sin \frac{z}{2}$$

$$(D) \sin^2 \left(\frac{X+Y}{2} \right) = \cos^2 \frac{Z}{2} = \frac{S(S-z)}{xy} = \frac{9 \cdot 2}{5 \cdot 6} = \frac{3}{5}$$

8*. In a triangle PQR, let $\angle PQR = 30^\circ$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE?

(A) $\angle QPR = 45^\circ$

[JEE(Advanced) 2018, Paper-1, (4, -2)/60]

(B) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120^\circ$

(C) The radius of the incircle of the triangle PQR is $10\sqrt{3} - 15$

(D) The area of the circumcircle of the triangle PQR is 100π

एक त्रिभुज (triangle) PQR में, माना कि $\angle PQR = 30^\circ$ और भुजाओं PQ और QR की लम्बाइयाँ क्रमशः $10\sqrt{3}$ और 10 हैं। तब निम्नलिखित में से कौनसा (से) कथन सत्य है (हैं) ?

(A) $\angle QPR = 45^\circ$

(B) त्रिभुज PQR का क्षेत्रफल (area) $25\sqrt{3}$ है और $\angle QRP = 120^\circ$

(C) त्रिभुज PQR के अंतर्वृत्त (incircle) की त्रिज्या (radius) $10\sqrt{3} - 15$ है।

(D) त्रिभुज PQR के परिवृत्त (circumcircle) का क्षेत्रफल 100π है।

Ans. (BCD)

$$\text{Sol. } \cos Q = \frac{100 + 300 - (PR)^2}{2 \cdot 10 \cdot 10\sqrt{3}} \Rightarrow \frac{\sqrt{3}}{2} = \frac{100 + 300 - (PR)^2}{2 \cdot 10 \cdot 10\sqrt{3}}$$

$$300 = 400 - (PR)^2 \Rightarrow PR = 10$$

$$\Delta = \frac{1}{2} (PQ)(QR) \sin Q = \frac{1}{2} \cdot 10 \cdot 10\sqrt{3} \times \frac{1}{2} = 25\sqrt{3}$$

$$r = \frac{\Delta}{s} = \frac{25\sqrt{3} \times 2}{(20 + 10\sqrt{3})} = \frac{50\sqrt{3}}{20 + 10\sqrt{3}} = \frac{5\sqrt{3}}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}} = 5(2\sqrt{3} - 3) = 10\sqrt{3} - 15$$

$$\text{by sine rule ज्या नियम } \frac{10\sqrt{3}}{\sin R} = \frac{10}{\sin Q} \Rightarrow \angle R = 30^\circ$$

$$2(\text{circumradiusपरित्रिज्या}) = \frac{PR}{\sin Q} = \frac{10}{1/2} \Rightarrow \text{circumradiusपरित्रिज्या} = 10$$

$$\text{Hence area of circumcircle} = \pi R^2 = 100\pi$$

$$\text{अतः परिवृत्त का क्षेत्रफल} = \pi R^2 = 100\pi$$



9. In a non-right-angled triangle ΔPQR , Let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S , the perpendicular from P meets the side QR at E , and RS and PE intersect at O . If $p = \sqrt{3}$, $q = 1$, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ?

{Solution of Triangle [ST-RA]-T-305}

[JEE(Advanced) 2019, Paper-1, (4, -1)/62]

(A) Length of $RS = \frac{\sqrt{7}}{2}$

(B) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$

(C) Radius of incircle of $\Delta PQR = \frac{\sqrt{3}}{2} (2 - \sqrt{3})$ (D) Length of $OE = \frac{1}{6}$

एक असमकोणीय त्रिभुज (non-right-angled) ΔPQR के लिए, माना कि p, q, r क्रमशः कोण P, Q, R के सामने वाली भुजाओं की लम्बाइयाँ दर्शाये हैं। R से खींची गयी माध्यिका (median) भुजा PQ से S पर मिलती है, P से खींचा गया अभिलम्ब (perpendicular) भुजा QR से E पर मिलता है तथा RS और PE एक दूसरे को O पर काटती है। यदि $p = \sqrt{3}$, $q = 1$ और ΔPQR के परिवृत्त (circumcircle) की त्रिज्या (radius) 1 है, तब निम्न में से कौन सा (से) विकल्प सही है (हैं) ?

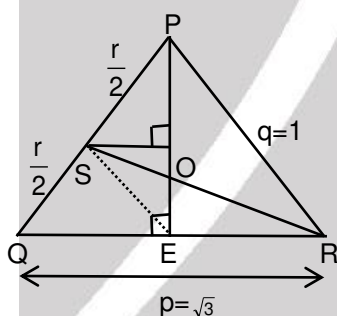
(A) RS की लम्बाई $= \frac{\sqrt{7}}{2}$

(B) ΔSOE का क्षेत्रफल (area) $= \frac{\sqrt{3}}{12}$

(C) ΔPQR के अंतर्वृत्त (incircle) की त्रिज्या $= \frac{\sqrt{3}}{2} (2 - \sqrt{3})$ (D) OE की लम्बाई $= \frac{1}{6}$

Ans. (ACD)

Sol.



$$\frac{p}{\sin P} = \frac{q}{\sin Q} = 2(1) \Rightarrow \sin P = \frac{\sqrt{3}}{2}, \sin Q = \frac{1}{2}$$

$$\Rightarrow \angle P = 60^\circ \text{ or } 120^\circ \text{ and और } \angle Q = 30^\circ \text{ or } 150^\circ$$

because $\angle P + \angle Q$ must be less than 180° but not equal to 90°

क्योंकि $\angle P + \angle Q$ का मान 180° से कम होगा परन्तु 90° के बराबर नहीं है।

$$\angle P = 120^\circ \text{ and और } \angle Q = 30^\circ \text{ and और } \angle R = 30^\circ \quad \frac{r}{\sin R} = 2 \Rightarrow r = 1$$





Now length of median माधिका की लम्बाई $RS = \frac{1}{2} \sqrt{2p^2 + 2q^2 - r^2} = \frac{1}{2} \sqrt{6+2-1} = \frac{\sqrt{7}}{2}$

⇒ option विकल्प (A) is correct सही है।

Inradius अंतःत्रिज्या $= \frac{2\Delta}{p+q+r} = \frac{\frac{2pqr}{4 \times (1)}}{p+q+r} = \frac{1}{2} \left(\frac{1 \times 1 \times \sqrt{3}}{1+1+\sqrt{3}} \right) = \frac{\sqrt{3}}{2} \left(\frac{2-\sqrt{3}}{1} \right) \Rightarrow$ option (C) is correct

⇒ $\frac{1}{2} \times \sqrt{3} \times PE = \frac{pqr}{4(1)}$ (equal area of Δ के क्षेत्रफल के बराबर)

⇒ $PE = \frac{1 \times 1 \times \sqrt{3}}{4} \times \frac{2}{\sqrt{3}} = \frac{1}{2}$

⇒ $OE = \frac{2(\text{Area of } \Delta OQR)}{QR} = \frac{2 \times \frac{1}{3} \left(\frac{1}{2} \cdot 1 \cdot \sqrt{3} \sin 30^\circ \right)}{\sqrt{3}} = \frac{1}{6}$

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

भाग - II : JEE (MAIN) / AIEEE (पिछले वर्षों) के प्रश्न

1. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A **false** statement among the following is [AIEEE - 2010 (4, -1), 144]

- (1) There is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$. (2*) There is a regular polygon with $\frac{r}{R} = \frac{2}{3}$.
 (3) There is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$. (4) There is a regular polygon with $\frac{r}{R} = \frac{1}{2}$.

एक समबहुभुज के लिए, माना अन्तःवृत्त तथा परिवृत्त की त्रिज्याएँ क्रमशः r तथा R है। निम्नलिखित में से कौनसा प्रकथन मिथ्या है ?

- (1) एक समबहुभुज ऐसा है जिसके लिए $\frac{r}{R} = \frac{1}{\sqrt{2}}$. (2*) एक समबहुभुज ऐसा है जिसके लिए $\frac{r}{R} = \frac{2}{3}$.
 (3) एक समबहुभुज ऐसा है जिसके लिए $\frac{r}{R} = \frac{\sqrt{3}}{2}$. (4) एक समबहुभुज ऐसा है जिसके लिए $\frac{r}{R} = \frac{1}{2}$.

Ans. (2)

Sol. $\frac{r}{R} = \cos \left(\frac{\pi}{n} \right)$

Let $\cos \frac{\pi}{n} = \frac{2}{3}$ for some $n \geq 3, n \in \mathbb{N}$

As $\frac{1}{2} < \frac{2}{3} < \frac{1}{\sqrt{2}} \Rightarrow \cos \frac{\pi}{3} < \cos \frac{\pi}{n} < \cos \frac{\pi}{4} \Rightarrow \frac{\pi}{3} > \frac{\pi}{n} > \frac{\pi}{4}$

⇒ $3 < n < 4$, which is not possible

so option (2) is the false statement

so it will be the right choice

Hence correct option is (2)

Hindi. $\frac{r}{R} = \cos \left(\frac{\pi}{n} \right)$

माना $\cos \frac{\pi}{n} = \frac{2}{3}$ किसी $n \geq 3, n \in \mathbb{N}$ के लिए





$$\text{चूँकि } \frac{1}{2} < \frac{2}{3} < \frac{1}{\sqrt{2}} \Rightarrow \cos \frac{\pi}{3} < \cos \frac{\pi}{n} < \cos \frac{\pi}{4} \Rightarrow \frac{\pi}{3} > \frac{\pi}{n} > \frac{\pi}{4}$$

$\Rightarrow 3 < n < 4$, जो संभव नहीं है।

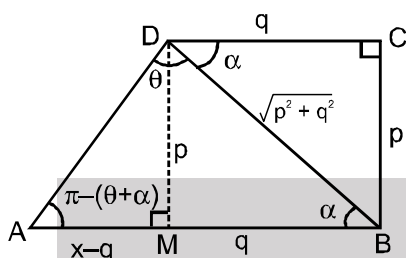
विकल्प (2), असत्य कथन है। अतः सही विकल्प (2) है।

2. ABCD is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, then AB is equal to : **[AIEEE - 2013, (4, -1), 120]**

ABCD एक ऐसा समलंब चतुर्भुज है जिसमें AB तथा CD समांतर हैं तथा $BC \perp CD$ है। यदि $\angle ADB = \theta$, $BC = p$ तथा $CD = q$, है, तो AB बराबर है : **[AIEEE - 2013, (4, -1/4), 360]**

(1*) $\frac{(p^2 + q^2) \sin \theta}{p \cos \theta + q \sin \theta}$ (2) $\frac{p^2 + q^2 \cos \theta}{p \cos \theta + q \sin \theta}$ (3) $\frac{p^2 + q^2}{p^2 \cos \theta + q^2 \sin \theta}$ (4) $\frac{(p^2 + q^2) \sin \theta}{(p \cos \theta + q \sin \theta)^2}$

Sol. (1)



Let (माना) $AB = x$

$$\tan(\pi - \theta - \alpha) = \frac{p}{x - q} \Rightarrow \tan(\theta + \alpha) = \frac{p}{q - x}$$

$$\Rightarrow q - x = p \cot(\theta + \alpha)$$

$$\Rightarrow x = q - p \cot(\theta + \alpha)$$

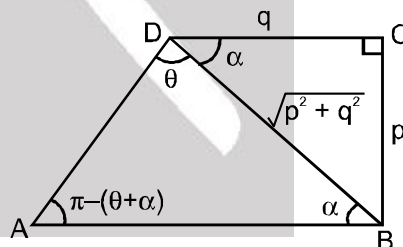
$$= q - p \left(\frac{\cot \theta \cot \alpha - 1}{\cot \alpha + \cot \theta} \right)$$

$$= q - p \left(\frac{\frac{q}{p} \cot \theta - 1}{\frac{q}{p} + \cot \theta} \right) = q - p \left(\frac{q \cot \theta - p}{q + p \cot \theta} \right) = q - p \left(\frac{q \cos \theta - p \sin \theta}{q \sin \theta + p \cos \theta} \right)$$

$$\Rightarrow x = \frac{q^2 \sin \theta + pq \cos \theta - pq \cos \theta + p^2 \sin \theta}{p \cos \theta + q \sin \theta} \Rightarrow AB = \frac{(p^2 + q^2) \sin \theta}{p \cos \theta + q \sin \theta}$$

Alternative

From Sine Rule ज्या नियम से



$$\frac{AB}{\sin \theta} = \frac{\sqrt{p^2 + q^2}}{\sin(\pi - (\theta + \alpha))}$$

$$AB = \frac{\sqrt{p^2 + q^2} \sin \theta}{\sin \theta \cos \alpha + \cos \theta \sin \alpha}$$

$$= \frac{(p^2 + q^2) \sin \theta}{q \sin \theta + p \cos \theta}$$

$$\left(\because \cos \alpha = \frac{q}{\sqrt{p^2 + q^2}} \right)$$



$$= \frac{(p^2 + q^2) \sin \theta}{p \cos \theta + q \sin \theta}$$

3. With the usual notation, in $\triangle ABC$, if $\angle A + \angle B = 120^\circ$, $a = \sqrt{3} + 1$ and $b = \sqrt{3} - 1$, then the ratio $\angle A : \angle B$, is:

सामान्य संकेतों में $\triangle ABC$ में यदि $\angle A + \angle B = 120^\circ$, $a = \sqrt{3} + 1$ तथा $b = \sqrt{3} - 1$ है, तो अनुपात $\angle A : \angle B$ बराबर है:

[JEE(Main) 2019, Online (10-01-19), P-2 (4, - 1), 120]

(1) 9 : 7

(2) 7 : 1

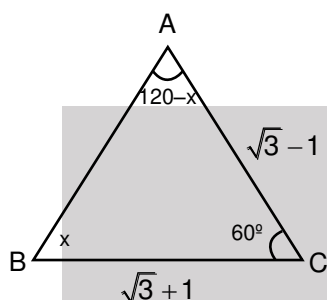
(3) 3 : 1

(4) 5 : 3

Ans. (2)

Sol.
$$\frac{\sqrt{3} + 1}{\sin(120 - x)} = \frac{\sqrt{3} - 1}{\sin x}$$

$$\frac{\sqrt{3} + 1}{\sqrt{3} - 1} = \frac{\sin(120^\circ - x)}{\sin x}$$



$$\frac{\sqrt{3} + 1}{\sqrt{3} - 1} - \frac{1}{2} = \frac{\sqrt{3}}{2} \cot x$$

$$\frac{3 + 2\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \cot x$$

$$\cot x = \sqrt{3} + 2$$

$$\tan x = 2 - \sqrt{3}$$

$$x = 15^\circ$$

$$120 - x = 105^\circ$$

$$\therefore \frac{\angle A}{\angle B} = \frac{7}{1} \quad (7 : 1)$$

4. In a triangle, the sum of lengths of two sides is x and the product of the lengths of the same two sides is y . If $x^2 - c^2 = y$, where c is the length of the third side of the triangle, then the circumradius of the triangle is

एक त्रिभुज की दो भुजाओं की लम्बाई का योग x है और इन्हीं दो भुजाओं की लम्बाई का गुणनफल y है। यदि $x^2 - c^2 = y$ जहाँ c त्रिभुज की तीसरी भुजा की लम्बाई है, तब त्रिभुज के परिवृत्त की त्रिज्या है -

[JEE(Main) 2019, Online (11-01-19), P-1 (4, - 1), 120]

(1) $\frac{c}{\sqrt{3}}$

(2) $\frac{3}{2} y$

(3) $\frac{c}{3}$

(4) $\frac{y}{\sqrt{3}}$

Ans. (1)

Sol. Let a, b, c be the three sides, given

त्रिभुज की भुजाएँ a, b, c

$$a + b = x, ab = y, (a + b)^2 - c^2 = ab$$



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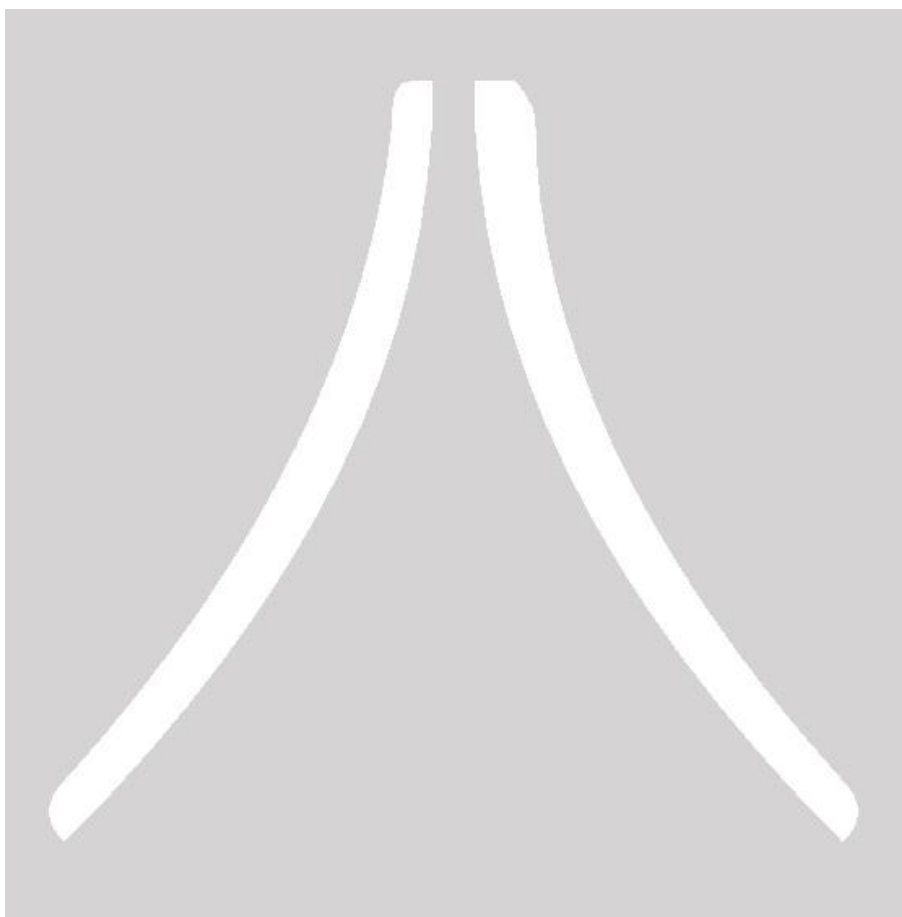
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Toll Free : 1800 258 5555 | CIN: U80302RJ2007PLC024029

ADVSOT - 51



here यहाँ $\frac{a^2 + b^2 - c^2}{2ab} = -\frac{1}{2} \Rightarrow \cos C = -\frac{1}{2}$

$$\frac{c}{\sin C} = 2R \Rightarrow \frac{2c}{\sqrt{3}} = 2R \Rightarrow R = \frac{c}{\sqrt{3}}$$





HLP Answers

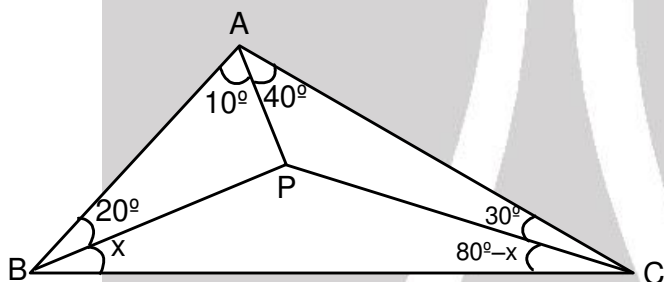
SUBJECTIVE QUESTIONS

विषयात्मक प्रश्न (SUBJECTIVE QUESTIONS)

This questions paste Staright Line sheets

1. In $\triangle ABC$, P is an interior point such that $\angle PAB = 10^\circ$, $\angle PBA = 20^\circ$, $\angle PCA = 30^\circ$, $\angle PAC = 40^\circ$ then prove that $\triangle ABC$ is isosceles
 $\triangle ABC$ में P आन्तरिक बिन्दु P इस प्रकार है कि $\angle PAB = 10^\circ$, $\angle PBA = 20^\circ$, $\angle PCA = 30^\circ$, $\angle PAC = 40^\circ$ तब सिद्ध कीजिए कि $\triangle ABC$ समद्विबाहु है।

Sol.



From $\triangle APB$, $\triangle PBC$ and $\triangle PCA$, using sine rule

$$\frac{AP}{\sin 20} = \frac{BP}{\sin 10}$$

$$\frac{BP}{\sin(80-x)} = \frac{PC}{\sin x}$$

$$\frac{PC}{\sin 40} = \frac{AP}{\sin 30}$$

$$\frac{AP}{\sin \angle ABP} \cdot \frac{BP}{\sin \angle PCB} \cdot \frac{CP}{\sin \angle PAC} = \frac{AP}{\sin \angle PCA} \cdot \frac{BP}{\sin \angle PBC} \cdot \frac{CP}{\sin \angle PAB}$$

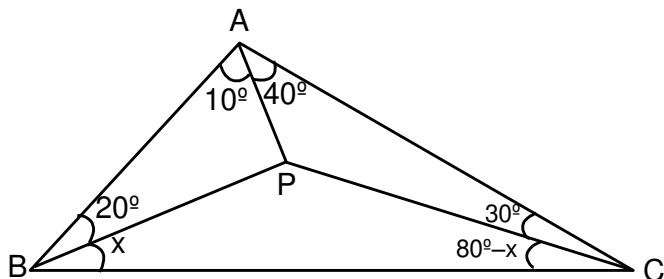
$$\Rightarrow \sin 30^\circ \cdot \sin x \cdot \sin 10^\circ = \sin 20^\circ \cdot \sin(80^\circ - x) \cdot \sin 40^\circ$$

$$\Rightarrow x = 60^\circ$$

$$\therefore \angle BCA = \angle CAB = 50^\circ$$

So, $\triangle ABC$ is an isosceles triangle.

Sol.



$\triangle APB$, $\triangle PBC$ और $\triangle PCA$, ज्या नियम से

$$\frac{AP}{\sin 20} = \frac{BP}{\sin 10}$$

$$\frac{BP}{\sin(80-x)} = \frac{PC}{\sin x}$$





$$\frac{PC}{\sin 40^\circ} = \frac{AP}{\sin 30^\circ}$$

$$\frac{AP}{\sin \angle ABP} \cdot \frac{BP}{\sin \angle PCB} \cdot \frac{CP}{\sin \angle PAC} = \frac{AP}{\sin \angle PCA} \cdot \frac{BP}{\sin \angle PBC} \cdot \frac{CP}{\sin \angle PAB}$$

$$\Rightarrow \sin 30^\circ \cdot \sin x \cdot \sin 10^\circ = \sin 20^\circ \cdot \sin(80^\circ - x) \cdot \sin 40^\circ$$

$$\Rightarrow x = 60^\circ$$

$$\therefore \angle BCA = \angle CAB = 50^\circ$$

इसलिए $\triangle ABC$ समद्विबाहु त्रिभुज है।

2. In a triangle ABC, if $a \tan A + b \tan B = (a + b) \tan \left(\frac{A+B}{2} \right)$, prove that triangle is isosceles.

त्रिभुज ABC में, यदि $a \tan A + b \tan B = (a + b) \tan \left(\frac{A+B}{2} \right)$ हो, तो सिद्ध कीजिए कि त्रिभुज समद्विबाहु है।

Sol. $\therefore a \tan A + b \tan B = (a + b) \tan \left(\frac{A+B}{2} \right)$

$$\Rightarrow a \left[\tan A - \tan \left(\frac{A+B}{2} \right) \right] = b \left[\tan \left(\frac{A+B}{2} \right) - \tan B \right]$$

$$\Rightarrow a \left[\frac{\sin A \cos \left(\frac{A+B}{2} \right) - \sin \left(\frac{A+B}{2} \right) \cos A}{\cos A \cos \left(\frac{A+B}{2} \right)} \right] = b \frac{\sin \left(\frac{A+B}{2} - B \right)}{\cos B \cos \left(\frac{A+B}{2} \right)}$$

$$\Rightarrow \frac{a \sin \left(A - \frac{A+B}{2} \right)}{\cos A \cos \left(\frac{A+B}{2} \right)} = \frac{b \sin \left(\frac{A-B}{2} \right)}{\cos B \cos \left(\frac{A+B}{2} \right)}$$

$$\Rightarrow \sin \left(\frac{A-B}{2} \right) \left(\frac{a}{\cos A} - \frac{b}{\cos B} \right) = 0$$

$$\Rightarrow \sin \left(\frac{A-B}{2} \right) = 0 \quad \text{or या} \quad \frac{a}{\cos A} - \frac{b}{\cos B} = 0$$

$$\Rightarrow A = B \quad \text{or या} \quad 2R (\tan A - \tan B) = 0$$

$$\Rightarrow \tan A = \tan B$$

$$\Rightarrow A = B$$

3. In any triangle ABC, if $2\Delta a - b^2c = c^3$, (where Δ is the area of triangle), then prove that $\angle A$ is obtuse
 किसी त्रिभुज ABC में यदि $2\Delta a - b^2c = c^3$, (जहाँ Δ त्रिभुज का क्षेत्रफल है) तब सिद्ध कीजिए $\angle A$ अधिक कोण है।

Sol. $2\Delta a - b^2c = c^3$

$$\Rightarrow 2\Delta a^2b = abc(b^2 + c^2)$$

$$\Rightarrow \frac{(a^2b)abc}{2R} = abc(b^2 + c^2)$$

$$\Rightarrow \frac{a^2b}{2R} = b^2 + c^2$$

$$\Rightarrow a^2 \sin B = b^2 + c^2$$

If $\sin B = 1$, then $a^2 = b^2 + c^2$, which is not possible

$$\therefore \sin B \neq 1$$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 \sin B - a^2}{2bc}$$

$$< 0$$

$\therefore A$ is obtuse





Sol. $2\Delta a - b^2c = c^3$
 $\Rightarrow 2\Delta a^2b = abc(b^2 + c^2)$
 $\Rightarrow \frac{(a^2b)abc}{2R} = abc(b^2 + c^2)$
 $\Rightarrow \frac{a^2b}{2R} = b^2 + c^2$
 $\Rightarrow a^2 \sin B = b^2 + c^2$
 यदि $\sin B = 1$, तब $a^2 = b^2 + c^2$, जो कि संभव नहीं है।
 $\therefore \sin B \neq 1$
 $\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 \sin B - a^2}{2bc}$
 < 0
 $\therefore A$ अधिक कोण है।

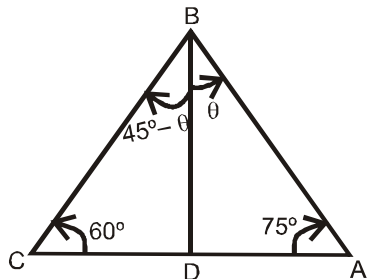
4. If in a triangle ABC, $\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$ prove that the triangle ABC is either isosceles or right angled.

यदि त्रिभुज ABC में $\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$ हो, तो सिद्ध कीजिए कि त्रिभुज ABC या तो समद्विबाहु त्रिभुज है या समकोण त्रिभुज है।

Sol. $\cos A(\sin B - \sin C) + (\sin 2B - \sin 2C) = 0$
 $\Rightarrow \cos A(\sin B - \sin C) + 2 \cos(B+C) \sin(B-C) = 0$ $\therefore B+C = \pi - A$
 $\Rightarrow \cos A(\sin B - \sin C) - 2 \cos A \sin(B-C) = 0$
 $\Rightarrow \cos A[(\sin B - \sin C) - 2(\sin B \cos C - \cos B \sin C)] = 0$
 $\Rightarrow$ either या तो $\cos A = 0 \Rightarrow A = 90^\circ \Rightarrow$ right angled समकोण त्रिभुज होगा
 or या $(\sin B - \sin C) - 2(\sin B \cos C - \cos B \sin C) = 0$
 $\Rightarrow (b-c) - 2\left(b \cdot \frac{a^2 + b^2 - c^2}{2ab} - c \cdot \frac{a^2 + c^2 - b^2}{2ac}\right) = 0$
 $\Rightarrow a(b-c) - 2(b^2 - c^2) = 0$
 $(b-c)[a - 2(b+c)] = 0$
 $\therefore b-c=0 \Rightarrow b=c \Rightarrow$ isosceles अतः त्रिभुज समद्विबाहु त्रिभुज होगा।

5. In a ΔABC , $\angle C = 60^\circ$ and $\angle A = 75^\circ$. If D is a point on AC such that the area of the ΔBAD is $\sqrt{3}$ times the area of the ΔBCD , find the $\angle ABD$.
 त्रिभुज ABC में $\angle C = 60^\circ$ एवं $\angle A = 75^\circ$ है। यदि भुजा AC पर एक बिन्दु D इस प्रकार है कि ΔBAD का क्षेत्रफल, त्रिभुज BCD के क्षेत्रफल का $\sqrt{3}$ गुना है। $\angle ABD$ का मान ज्ञात कीजिए।

Ans. $\angle ABD = 30^\circ$



Sol.

Area of $\Delta BAD = \sqrt{3} \times$ Area of ΔBCD ΔBAD का क्षेत्रफल $= \sqrt{3} \times \Delta BCD$ का क्षेत्रफल
 $\Rightarrow \frac{1}{2} BD \times BA \sin \theta = \sqrt{3} \times \frac{1}{2} BC \times BD \sin (45^\circ - \theta)$





$$\frac{BA}{BC} = \sqrt{3} \frac{\sin(45^\circ - \theta)}{\sin \theta} \dots\dots\dots(1)$$

From Sine-Rule ज्या नियम का प्रयोग करने पर

$$\frac{BC}{\sin 75^\circ} = \frac{AB}{\sin 60^\circ}$$

$$\therefore \frac{BA}{BC} = \frac{\sin 60^\circ}{\sin 75^\circ} = \frac{\sqrt{3}\sqrt{2}}{\sqrt{3}+1}$$

$\therefore$ From equation (1) समीकरण (1) से

$$\frac{\sqrt{3}\sqrt{2}}{(\sqrt{3}+1)} = \sqrt{3} \left[\frac{1}{\sqrt{2}} \cot \theta - \frac{1}{\sqrt{2}} \right]$$

$$\Rightarrow \frac{2}{(\sqrt{3}+1)} = \cot \theta - 1 \Rightarrow \frac{2(\sqrt{3}-1)}{2} = \cot \theta - 1$$

$$\Rightarrow \cot \theta = \sqrt{3} \Rightarrow \theta = 30^\circ \Rightarrow \angle ABD = 30^\circ$$

6. In a $\triangle ABC$, if a, b and c are in A.P., prove that $\cos A \cot \frac{A}{2}$, $\cos B \cot \frac{B}{2}$, and $\cos C \cot \frac{C}{2}$ are in A.P.

यदि किसी त्रिभुज ABC में a, b एवं c समान्तर श्रेढ़ी में हो, तो सिद्ध कीजिए कि $\cos A \cot \frac{A}{2}$, $\cos B \cot \frac{B}{2}$ एवं $\cos C \cot \frac{C}{2}$ समान्तर श्रेढ़ी में है।

Sol. $\therefore \cos A \cot \frac{A}{2} = \left(1 - 2\sin^2 \frac{A}{2}\right) \frac{\cos \frac{A}{2}}{\sin \frac{A}{2}} = \cot \frac{A}{2} - \sin A$

Similarly $\cos B \cot \frac{B}{2} = \cot \frac{B}{2} - \sin B$

and $\cos C \cot \frac{C}{2} = \cot \frac{C}{2} - \sin C$

$\therefore a, b, c$ are in A.P.

$\therefore \sin A, \sin B, \sin C$ are also in A.P.

$\therefore a, b, c$ are in A.P.

$\therefore \cot \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2}$ are also in A.P.

$\therefore \cot \frac{A}{2} - \sin A, \cot \frac{B}{2} - \sin B, \cot \frac{C}{2} - \sin C$ are also in A.P.

Hindi. $\therefore \cos A \cot \frac{A}{2} = \left(1 - 2\sin^2 \frac{A}{2}\right) \frac{\cos \frac{A}{2}}{\sin \frac{A}{2}} = \cot \frac{A}{2} - \sin A$

इसी प्रकार $\cos B \cot \frac{B}{2} = \cot \frac{B}{2} - \sin B$

और $\cos C \cot \frac{C}{2} = \cot \frac{C}{2} - \sin C$

$\therefore a, b, c$ समान्तर श्रेढ़ी में है।

$\therefore \sin A, \sin B, \sin C$ भी समान्तर श्रेढ़ी में होंगे।





$\therefore a, b, c$ समान्तर श्रेढ़ी में है।

$\therefore \cot \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2}$ भी समान्तर श्रेढ़ी में होंगे।

$\therefore \cot \frac{A}{2} - \sin A, \cot \frac{B}{2} - \sin B, \cot \frac{C}{2} - \sin C$ भी समान्तर श्रेढ़ी में होंगे।

7. In a triangle ABC, prove that the area of the incircle is to the area of triangle itself is,
 $\pi : \cot \left(\frac{A}{2} \right) \cdot \cot \left(\frac{B}{2} \right) \cdot \cot \left(\frac{C}{2} \right)$.

त्रिभुज ABC में सिद्ध कीजिए कि अन्तःवृत्त के क्षेत्रफल और स्वयं इस त्रिभुज के क्षेत्रफल का अनुपात

$\pi : \cot \left(\frac{A}{2} \right) \cdot \cot \left(\frac{B}{2} \right) \cdot \cot \left(\frac{C}{2} \right)$ है।

Sol. $\therefore \frac{\text{Area of incircle}}{\text{Area of } \triangle ABC} = \frac{\text{अन्तः वृत्त का क्षेत्रफल}}{\triangle ABC \text{ का क्षेत्रफल}} = \frac{\pi r^2}{\frac{1}{2}bc \sin A}$

$$= \frac{\pi \times 16R^2 \times \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2}}{\frac{1}{2}(2R \sin B)(2R \sin C) \left(2 \sin \frac{A}{2} \cos \frac{A}{2} \right)} = \frac{4\pi \sin \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2}}{\left(2 \sin \frac{B}{2} \cos \frac{B}{2} \right) \left(2 \sin \frac{C}{2} \cos \frac{C}{2} \right) \cos \frac{A}{2}}$$

$$= \frac{\pi \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{\pi}{\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}} = \pi : \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

8. In $\triangle ABC$, prove that $a^2(s-a) + b^2(s-b) + c^2(s-c) = 4R\Delta \left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right)$
 $\triangle ABC$, में सिद्ध कीजिए $a^2(s-a) + b^2(s-b) + c^2(s-c) = 4R\Delta \left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right)$

Sol. L.H.S. = $\frac{1}{2} [a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c)]$

$$= \frac{1}{2} [a(b^2 + c^2 - a^2) + b(c^2 + a^2 - b^2) + c(a^2 + b^2 - c^2)]$$

$$= \frac{1}{2} (2abc \cos A + 2abc \cos B + 2abc \cos C)$$

$$= abc \left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right)$$

$$= 4R\Delta \left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right)$$

9. In any $\triangle ABC$, prove that
 त्रिभुज ABC में सिद्ध कीजिए कि -

(i) $(r_3 + r_1)(r_3 + r_2) \sin C = 2r_3 \sqrt{r_2 r_3 + r_3 r_1 + r_1 r_2}$

(ii) $\frac{\tan \frac{A}{2}}{(a-b)(a-c)} + \frac{\tan \frac{B}{2}}{(b-a)(b-c)} + \frac{\tan \frac{C}{2}}{(c-a)(c-b)} = \frac{1}{\Delta}$

(iii) $(r+r_1) \tan \frac{B-C}{2} + (r+r_2) \tan \frac{C-A}{2} + (r+r_3) \tan \frac{A-B}{2} = 0$





$$(iv) \quad r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - a^2 - b^2 - c^2.$$

Sol. (i) L.H.S. बायां पक्ष = $(r_3 + r_1)(r_3 + r_2) \sin C$

$$= \frac{\Delta b}{(s-a)(s-c)} \frac{\Delta a}{(s-c)(s-b)} \sin C$$

$$= \frac{ab\Delta^2}{(s-a)(s-b)(s-c)(s-c)} \sin C$$

$$= \frac{ab}{(s-a)(s-b)(s-c)(s-c)} s(s-a)(s-b)(s-c) \sin C$$

$$= \frac{abs \sin C}{(s-c)}$$

$$= \frac{2\Delta s}{(s-c)} = 2sr_3 \sqrt{r_2r_3 + r_3r_1 + r_1r_2}$$

R.H.S. दायां पक्ष = $2r_3$

$$= 2r_3 \sqrt{s^2} = 2sr_3$$

$\therefore$ L.H.S. बायां पक्ष = R.H.S. दायां पक्ष

(ii) L.H.S. बायां पक्ष = $-\frac{1}{\Delta} \left[\frac{(s-b)(s-c)}{(a-b)(c-a)} + \frac{(s-a)(s-c)}{(a-b)(b-c)} + \frac{(s-a)(s-b)}{(c-a)(b-c)} \right]$

$$= -\frac{1}{\Delta} \left[\frac{(s-b)(s-c)(b-c) + (s-a)(s-c)(c-a) + (s-a)(s-b)(a-b)}{(a-b)(b-c)(c-a)} \right]$$

$$= \frac{1}{\Delta} = \text{R.H.S. दायां पक्ष}$$

(iii) First term (प्रथम पद) = $(r + r_1) \tan \frac{B-C}{2}$

$$= \left(\frac{\Delta}{s} + \frac{\Delta}{s-a} \right) \left(\frac{b-c}{b+c} \right) \cot \frac{A}{2}$$

$$= \frac{\Delta(2s-a)}{s(s-a)} \cdot \left(\frac{b-c}{b+c} \right) \cdot \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= b-c$$

similarly second term (इसी प्रकार द्वितीय पद) = $c-a$

& third term (और तृतीय पद) = $a-b$

$\therefore$ L.H.S. बायां पक्ष = $b-c + c-a + a-b = 0 = \text{R.H.S. दायां पक्ष}$

(iv) $\therefore r_1 + r_2 + r_3 - r = 4R$

$\therefore (r_1 + r_2 + r_3 - r)^2 = r_1^2 + r_2^2 + r_3^2 + r^2 - 2r(r_1 + r_2 + r_3) + 2(r_1r_2 + r_2r_3 + r_3r_1)$ (i)

$\therefore r(r_1 + r_2 + r_3) = ab + bc + ca - s^2$

and (और) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

$\therefore$ from equation (i) समीकरण (i) से

$$16R^2 = r^2 + r_1^2 + r_2^2 + r_3^2 - 2(ab + bc + ca - s^2) + 2s^2$$

$\therefore r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - 4s^2 + 2(ab + bc + ca)$

$$= 16R^2 - (a+b+c)^2 + 2(ab + bc + ca)$$

$$= 16R^2 - a^2 - b^2 - c^2$$



10. In an acute angled triangle ABC, $r + r_1 = r_2 + r_3$ and $\angle B > \frac{\pi}{3}$, then prove that $b + 3c < 3a < 3b + 3c$

न्यूनकोण त्रिभुज ABC में $r + r_1 = r_2 + r_3$ और $\angle B > \frac{\pi}{3}$, तब सिद्ध कीजिए $b + 3c < 3a < 3b + 3c$

Sol. $r - r_2 = r_3 - r_1 \Rightarrow \frac{\Delta}{s} - \frac{\Delta}{s-b} = \frac{\Delta}{s-c} - \frac{\Delta}{s-a}$

or या $\frac{-b}{s(s-b)} = \frac{c-a}{(s-c)(s-a)}$

or या $\frac{(s-a)(s-c)}{s(s-b)} = \frac{a-c}{b} \Rightarrow \tan^2 \frac{B}{2} = \frac{a-c}{b}$

But परन्तु $\frac{B}{2} \in \left(\frac{\pi}{6}, \frac{\pi}{4}\right)$. Therefore इसलिए,

$\tan^2 \frac{B}{2} \in \left(\frac{1}{3}, 1\right) \Rightarrow \frac{1}{3} < \frac{a-c}{b} < 1$

or या $b < 3a - 3c < 3b$

$b + 3c < 3a < 3b + 3c$

11. If the inradius in a right angled triangle with integer sides is r . Prove that

(i) If $r = 4$, the greatest perimeter (in units) is 90

(ii) If $r = 5$, the greatest area (in sq. units) is 330

यदि समकोण त्रिभुज जिसकी भुजाएँ पूर्णांक हैं अन्तः त्रिज्या r है तब सिद्ध कीजिए कि

(i) यदि $r = 4$, तब अधिकतम परिमाप 90 (ईकाई में) है।

(ii) यदि $r = 5$ हो तो अधिकतम क्षेत्रफल (ईकाई वर्ग में) 330 है।

Sol. (i and ii) Let a, b and c ($a < b < c$) be the sides of given triangle.

Also, $2r = a + b - c$

When $r = 4$ then, $(a, b) = (9, 40), (10, 24), (12, 16)$

$\therefore$ Greatest perimeter = $9 + 40 + 41 = 90$ units

when $r = 5$ then $(a, b) = (11, 60), (12, 35), (15, 20)$

$\therefore$ Greatest area = $\frac{11 \times 60}{2} = 330$ sq. unit

Sol. (i और ii) माना a, b और c ($a < b < c$) त्रिभुज की भुजाएँ

तथा, $2r = a + b - c$

जब $r = 4$ तब, $(a, b) = (9, 40), (10, 24), (12, 16)$

$\therefore$ अधिकतम परिमाप = $9 + 40 + 41 = 90$ units

जहाँ $r = 5$ तब $(a, b) = (11, 60), (12, 35), (15, 20)$

$\therefore$ अधिकतम क्षेत्रफल = $\frac{11 \times 60}{2} = 330$ वर्ग इकाई

12. If $\left(1 - \frac{r_1}{r_2}\right) \left(1 - \frac{r_1}{r_3}\right) = 2$, then prove that the triangle is right angled.

यदि $\left(1 - \frac{r_1}{r_2}\right) \left(1 - \frac{r_1}{r_3}\right) = 2$ हो, तो सिद्ध कीजिए कि त्रिभुज, समकोण त्रिभुज है।

Sol. $\left(1 - \frac{r_1}{r_2}\right) \left(1 - \frac{r_1}{r_3}\right) = 2 \Rightarrow \left(1 - \frac{s-b}{s-a}\right) \left(1 - \frac{s-c}{s-a}\right) = 2$

$\Rightarrow \frac{(b-a)(c-a)}{(s-a)^2} = 2 \Rightarrow 2(bc - ab - ac + a^2) = (2s - 2a)^2$

$\Rightarrow 2bc - 2ab - 2ca + 2a^2 = (b^2 + c^2 + a^2 - 2ab + 2bc - 2ca)$
 $a^2 = b^2 + c^2$





$\Rightarrow$ triangle is right angled. त्रिभुज, समकोण त्रिभुज होगा।

13. DEF is the triangle formed by joining the points of contact of the incircle with the sides of the triangle ABC; prove that

(i) its sides are $2r \cos \frac{A}{2}$, $2r \cos \frac{B}{2}$ and $2r \cos \frac{C}{2}$,

(ii) its angles are $\frac{\pi}{2} - \frac{A}{2}$, $\frac{\pi}{2} - \frac{B}{2}$ and $\frac{\pi}{2} - \frac{C}{2}$

and

(iii) its area is $\frac{2\Delta^3}{(abc)s}$, i.e. $\frac{1}{2} r R \Delta$.

त्रिभुज ABC की भुजाओं को अन्तःवृत्त जिन बिन्दुओं पर स्पर्श करता है उनको मिलाने से त्रिभुज DEF निर्मित होता है। सिद्ध कीजिए कि -

(i) इसकी भुजाएँ $2r \cos \frac{A}{2}$, $2r \cos \frac{B}{2}$ and $2r \cos \frac{C}{2}$ हैं।

(ii) इसके कोण $\frac{\pi}{2} - \frac{A}{2}$, $\frac{\pi}{2} - \frac{B}{2}$ and $\frac{\pi}{2} - \frac{C}{2}$ हैं।

और

(iii) इसका क्षेत्रफल $\frac{2\Delta^3}{(abc)s}$ अर्थात् $\frac{1}{2} r R \Delta$ है।

Sol. (i) EIFA is a cyclic quadrilateral

$$\therefore \frac{EF}{\sin A} = AI$$

$$\therefore AI = r \operatorname{cosec} A/2$$

$$\therefore EF = r \operatorname{cosec} A/2 \cdot \sin A = 2r \cos A/2$$

similarly $DF = 2r \cos B/2$
and $DE = 2r \cos C/2$.

(ii) IECD is a cyclic quadrilateral

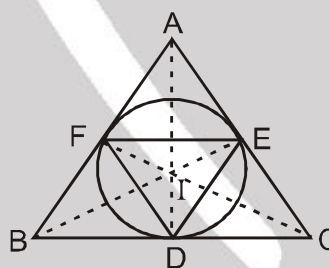
$$\therefore \angle ICE = \angle IDE = \frac{C}{2}$$

similarly $\angle IDF = \angle IBF = \frac{B}{2}$

$$\therefore \angle FDE = \frac{B}{2} + \frac{C}{2} = \frac{\pi - A}{2} = \frac{\pi}{2} - \frac{A}{2}$$

(iii) area of $\triangle DEF = \frac{1}{2} FD \cdot DE \sin \angle FDE$

$$= \frac{1}{2} \left(2r \cos \frac{B}{2} \right) \left(2r \cos \frac{C}{2} \right) \sin \left(\frac{\pi}{2} - \frac{A}{2} \right)$$





$$\begin{aligned}
 &= 2r^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \\
 &= 2r^2 \left(\frac{\sin A + \sin B + \sin C}{4} \right) \\
 &= \frac{r^2}{2} \left(\frac{2\Delta}{bc} + \frac{2\Delta}{ca} + \frac{2\Delta}{ab} \right) \\
 &= \frac{r^2}{2} \left[\frac{2\Delta (a+b+c)}{abc} \right] = \frac{r^2 \Delta \cdot 2s}{abc} \\
 &= \frac{2 r^2 \cdot \Delta s^2}{(abc)s} = \frac{2\Delta (r \cdot s)^2}{(abc)s} \\
 &= \frac{2\Delta^3}{(abc)s} = \frac{1}{2} \frac{r\Delta}{R}
 \end{aligned}$$

Hindi (i) EIFA एक चक्रीय चतुर्भुज है

$$\begin{aligned}
 \therefore \frac{EF}{\sin A} &= AI \\
 \therefore AI &= r \operatorname{cosec} A/2 \\
 \therefore EF &= r \operatorname{cosec} A/2 \sin A \\
 &= 2r \cos A/2
 \end{aligned}$$

इसी प्रकार, $DF = 2r \cos B/2$

और $DE = 2r \cos C/2$.

(ii) IECD एक चक्रीय चतुर्भुज है

$$\therefore \angle ICE = \angle IDE = \frac{C}{2}$$

इसी प्रकार $\angle IDF = \angle IBF = \frac{B}{2}$

$$\begin{aligned}
 \therefore \angle FDE &= \frac{B}{2} + \frac{C}{2} = \frac{\pi - A}{2} \\
 &= \frac{\pi}{2} - \frac{A}{2}
 \end{aligned}$$

(iii) $\triangle DEF$ का क्षेत्रफल $= \frac{1}{2} FD \cdot DE \sin \angle FDE$

$$\begin{aligned}
 &= \frac{1}{2} \left(2r \cos \frac{B}{2} \right) \left(2r \cos \frac{C}{2} \right) \sin \left(\frac{\pi}{2} - \frac{A}{2} \right) \\
 &= 2r^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \\
 &= 2r^2 \left(\frac{\sin A + \sin B + \sin C}{4} \right) \\
 &= \frac{r^2}{2} \left(\frac{2\Delta}{bc} + \frac{2\Delta}{ca} + \frac{2\Delta}{ab} \right) \\
 &= \frac{r^2}{2} \left[\frac{2\Delta (a+b+c)}{abc} \right] = \frac{r^2 \Delta \cdot 2s}{abc} \\
 &= \frac{2 r^2 \cdot \Delta s^2}{(abc)s} = \frac{2\Delta (r \cdot s)^2}{(abc)s} \\
 &= \frac{2\Delta^3}{(abc)s} = \frac{1}{2} \frac{r\Delta}{R}
 \end{aligned}$$

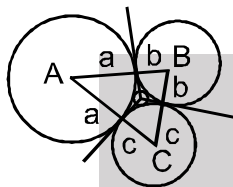




14. Three circles, whose radii are a , b and c , touch one another externally and the tangents at their points of contact meet in a point, prove that the distance of this point from either of their points of contact is

$$\left(\frac{abc}{a+b+c} \right)^{\frac{1}{2}}.$$

तीन वृत्त जिनकी त्रिज्याएँ a , b , c हैं, एक दूसरे को बाह्य स्पर्श करते हैं तथा उनके स्पर्श बिन्दुओं पर स्पर्श रेखाएँ एक बिन्दु पर मिलती हैं। सिद्ध कीजिए कि इस बिन्दु की उनके किसी स्पर्श बिन्दु से दूरी $\left(\frac{abc}{a+b+c} \right)^{\frac{1}{2}}$ है।



Sol.

required distance = inradius of ΔABC

अभीष्ट दूरी = ΔABC की अन्तःत्रिज्या

$$\therefore 2s = a + b + b + c + c + a$$

$$= 2(a + b + c)$$

$$s = a + b + c$$

$$\Delta = \sqrt{s(s-(a+b))(s-(b+c))(s-(c+a))} = \sqrt{(a+b+c)(abc)}$$

$\therefore$ required distance अभीष्ट दूरी

$$= \frac{\Delta}{s} = \frac{\sqrt{(a+b+c)(abc)}}{(a+b+c)} = \sqrt{\frac{abc}{a+b+c}} = \left(\frac{abc}{a+b+c} \right)^{\frac{1}{2}}$$

15. OA and OB are the equal sides of an isosceles triangle lying in the first quadrant making angles θ and ϕ respectively with x-axis. Show that the gradient of the bisector of acute angle AOB is $\operatorname{cosec} \beta - \cot \beta$ where $\beta = \phi + \theta$. (Where O is origin)

OA और OB समद्विबाहु त्रिभुज की दो बराबर भुजाएँ हैं जो प्रथम चतुर्थांश में क्रमशः x के साथ θ और ϕ कोण बनाती हैं। दर्शाइये कि न्यूनकोण AOB के अर्धक की प्रवणता $\operatorname{cosec} \beta - \cot \beta$ है जहाँ $\beta = \phi + \theta$. (जहाँ O मूल बिन्दु है)

Sol.

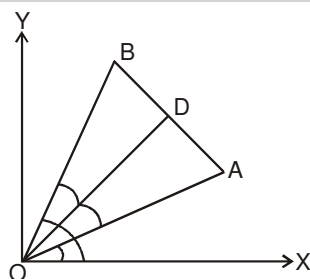
From the fig.

$$\angle AOD = \alpha = \angle DOB$$

$$\phi = \theta = 2\alpha$$

$$\text{or, } \alpha = \frac{\phi - \theta}{2}$$

$$\text{or, } \angle DOX = \theta + \alpha = \frac{\theta + \phi}{2}$$



$$\text{The gradient of OD} = \tan \frac{\theta + \phi}{2} = \frac{\sin \frac{\theta + \phi}{2}}{\cos \frac{\theta + \phi}{2}}$$

$$= \frac{2 \sin^2 \frac{\theta + \phi}{2}}{2 \cos \frac{\theta + \phi}{2} \sin \frac{\theta + \phi}{2}} = \frac{1 + \cos(\theta + \phi)}{\sin(\theta + \phi)} = \operatorname{cosec} \beta - \cot \beta.$$





16. The hypotenuse $BC = a$ of a right-angled triangle ABC is divided into n equal segments where n is odd. The segment containing the midpoint of BC subtends angle α at A . Also h is the altitude of the triangle through A . Prove that $\tan \alpha = \frac{4nh}{a(n^2 - 1)}$.

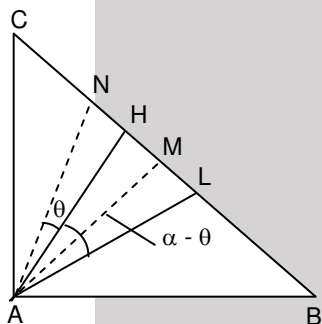
समकोण त्रिभुज ABC के कर्ण $BC = a$ को n बराबर खण्डों में विभाजित किया जाता है जहाँ n विषम है। BC के मध्य बिन्दु को रखने वाला खण्ड, A पर कोण α बनाती है तथा h , A से जाने वाला शीर्षलम्ब है तब सिद्ध कीजिए

$$\tan \alpha = \frac{4nh}{a(n^2 - 1)}$$

Sol. Let LN be the segment of the side BC containing its midpoint M . We have $BM = MC = AM = \frac{a}{2}$. Let

AH be the altitude from A on BC , with $AH = h$. Also $\angle LAN = \alpha$. Let $\angle NAH = \theta \Rightarrow \angle HAL = \alpha - \theta$.

From $\triangle AMH$, we have $MH = \sqrt{\frac{a^2}{4} - h^2}$. Also $LM = MN = \frac{a}{2n}$



$$\begin{aligned} \text{Now } \tan \alpha &= \tan(\alpha - \theta + \theta) = \frac{\tan(\alpha - \theta) + \tan \theta}{1 - \tan \theta \tan(\alpha - \theta)} = \frac{\frac{LH}{h} + \frac{NH}{h}}{1 - \frac{LH}{h} \cdot \frac{NH}{h}} \\ &= \frac{h(LH + NH)}{h^2 - LH \cdot NH} = \frac{h \cdot a}{n \left[h^2 - \left(\frac{a}{2n} + \sqrt{\frac{a^2}{4} - h^2} \right) \left(\frac{a}{2n} - \sqrt{\frac{a^2}{4} - h^2} \right) \right]} \\ &= \frac{ha}{n \left[h^2 - \frac{a^2}{4n^2} + \frac{a^2}{4} - h^2 \right]} = \frac{4nh}{a(n^2 - 1)}. \end{aligned}$$

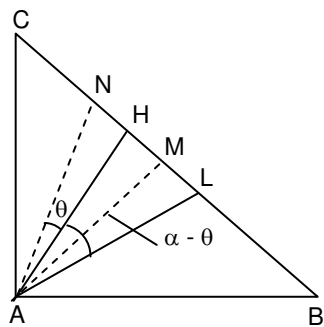
Hindi माना LN भुजा BC का खण्ड है जो इसके मध्य बिन्दु M को रखता है।

यहाँ $BM = MC = AM = \frac{a}{2}$.

माना AH , A से BC पर शीर्षलम्ब, $AH = h$. तथा $\angle LAN = \alpha$. माना $\angle NAH = \theta \Rightarrow \angle HAL = \alpha - \theta$.

$\triangle AMH$ से यहाँ $MH = \sqrt{\frac{a^2}{4} - h^2}$. इसलिए $LM = MN = \frac{a}{2n}$





$$\begin{aligned}
 \text{अब } \tan \alpha &= \tan(\alpha - \theta + \theta) = \frac{\tan(\alpha - \theta) + \tan \theta}{1 - \tan \theta \tan(\alpha - \theta)} = \frac{\frac{LH}{h} + \frac{NH}{h}}{1 - \frac{LH}{h} \cdot \frac{NH}{h}} \\
 &= \frac{h(LH + NH)}{h^2 - LH \cdot NH} = \frac{h \cdot a}{n \left[h^2 - \left(\frac{a}{2n} + \sqrt{\frac{a^2}{4} - h^2} \right) \left(\frac{a}{2n} - \sqrt{\frac{a^2}{4} - h^2} \right) \right]} \\
 &= \frac{ha}{n \left[h^2 - \frac{a^2}{4n^2} + \frac{a^2}{4} - h^2 \right]} = \frac{4nh}{a(n^2 - 1)}
 \end{aligned}$$

